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THE OCTOBER MEETING IN NEW YORK

The two hundred ninety-third meeting of the American Mathematical Society was held at Columbia University, on Saturday, October 31, 1931, extending through the usual morning and afternoon sessions. The attendance included the following one hundred members of the Society:

R. P. Agnew, R. L. Anderson, R. G. Archibald, J. W. Arnold, T. C. Benton, B. A. Bernstein, A. C. Berry, William James Berry, Joseph Bowden, A. B. Brown, J. H. Bushey, S. S. Cairns, B. H. Camp, G. A. Campbell, George Comenetz, L. D. Cummings, Jesse Douglas, Arnold Dresden, L. P. Eisenhart, W. W. Elliott, T. S. Fiske, W. B. Fite, D. A. Flanders, Tomlinson Fort, R. M. Foster, E. U. Frey, K. G. Fuller, C. A. Garabedian, H. M. Gehman, B. P. Gill, J. W. Glover, M. C. Gray, T. H. Gronwall, C. C. Grove, C. M. Hebbert, Robert Henderson, L. S. Hill, Einar Hille, Lulu Hofmann, Harold Hotelling, J. C. Hughes, L. W. Hussey, S. A. Joffe, R. A. Johnson, H. M. Kammerer, Edward Kasner, M. V. Kenny, J. R. Kline, M. S. Knebelman, B. O. Koopman, Mark Kormes, A. W. Landers, V. V. Latshaw, E. R. Lorch, James McGiffert, L. A. MacColl, F. H. Miller, E. C. Molina, Richard Morris, G. W. Mullins, C. A. Nelson, Oystein Ore, F. W. Owens, H. B. Owens, E. L. Post, A. G. Rau, H. W. Raudenbush, H. W. Reddick, M. S. Rees, R. G. D. Richardson, J. F. Ritt, H. P. Robertson, H. M. Schlauch, C. E. Seely, Wladimir Seidel, Edward Sellers, Stefan Serghiesco, J. A. Shohat, L. P. Siceloff, L. G. Simons, L. L. Smail, P. A. Smith, M. H. Stone, H. G. Swain, J. D. Tamarkin, Alexander Tartler, J. M. Thomas, M. M. Torrey, Annita Tuller, Oswald Veblen, M. E. Wells, Anna Pell Wheeler, H. S. White, Hassler Whitney, D. V. Widder, S. S. Wilks, W. A. Wilson, H. M. Wright, Jacob Yerushalmy, Leo Zippin.

At the meeting of the Council, the following eleven persons were elected to membership in the Society:

Mr. Ralph Alexander Beaver, New York State College, Albany;
Professor Gaston Swindell Bruton, University of the South;
Professor Montgomery Morton Culver, University of Pittsburgh;
Professor Mary de Lellis Gough (Sister Mary de Lellis), Incarnate Word College;
Mr. Ralph Aubrie Hefner, Georgia School of Technology;
Professor William Robert Hutcherson, Berea College;
Mr. Rufus Oldenburger, University of Michigan;
Mr. Theodore Paine Palmer, Harvard University;
Professor Mabel Frances Schmeiser, Nebraska State Teachers College, Wayne;
Dr. Rothwell Clifford Stephens, Knox College;
Mr. Yuan-Yung Tseng, Fellow of the Chinese Government.

The report of the Committee on the Cole Prize in the Theory of Numbers was received and the recipient chosen; it was voted

to announce the award at the Annual Meeting. A reciprocity agreement with the Deutsche Mathematiker-Vereinigung was consummated, details of which are given in the Notes in this Bulletin. An invitation was received from Harvard University to hold the winter meeting of 1933 at that institution. The difficult financial situation of the Society was discussed and various steps were approved with a view to ensuring that the main activities of the Society should not be too much hampered.

The following appointments were reported: to represent the Society at the inauguration of William Pearson Tolley as President of Allegheny College, October 9, 1931, Professor Virgil Snyder; to represent the Society at the inauguration of Harry Clinton Gossard as President of the New Mexico Normal University, October 8, 1931, Dean T. G. Rodgers; to represent the Society at the inauguration of James Monroe Smith as President of Louisiana State University, November 12, 1931, Professor H. E. Buchanan; to represent the Society at the meeting of the Rittenhouse Bicentenary Citizens Committee held in Philadelphia, October 12, 1931, Professor H. H. Mitchell.

At the request of the Program Committee, Professor D. V. Widder, of Harvard University, delivered, at the beginning of the afternoon session, an address entitled *Some recent developments in the theory of Laplace integrals.*

Titles and cross-references to the abstracts of the papers read at the regular sessions follow below. Dr. Hopf was introduced by Professor Tamarkin, Miss Sullivan by Professor Kellogg, and Mr. Winston by Professor Shohat. The papers whose abstract numbers are followed by the letter *t* were read by title.

1. *Covering of analytical loci by complexes*, by Dr. A. B. Brown and Professor B. O. Koopman. (Abstract No. 37–11–334.)

2. *On the double-point configurations of two projective planes on the same base as the result of the superposition of two distinct projectively related planes of euclidean space*, by Dr. Lulu Hofmann. (Abstract No. 37–11–335.)

3. *On the lines of Julia of entire functions*, Part I, by Dr. Wladimir Seidel and Dr. S. B. Littauer (National Research Fellows). (Abstract No. 37–11–336–*t*.)

4. *On the lines of Julia of entire functions*, Part II, by Dr. Wladimir Seidel and Dr. S. B. Littauer (National Research Fellows). (Abstract No. 37–11–337.)

5. *The condition for an orthonomic differential system*, by Professor J. M. Thomas. (Abstract No. 37–11–338.)

6. *A table of Fourier integrals for practical applications.* Preliminary report, by Mr. R. M. Foster. (Abstract No. 37–11–339.)

7. *The algebra of self-adjoint boundary value problems*, by Dr. V. V. Latshaw. (Abstract No. 37–11–340.)

8. *On interpolation*, by Professor J. A. Shohat. (Abstract No. 37–9–330.)

9. *On a generalized binomial series*, by Mr. E. A. Kholodovsky. (Abstract No. 37–7–256–*t*.)

10. *On mechanical quadratures of Gauss' type*, by Mr. C. Winston. (Abstract No. 37–9–333.)

11. *On a certain new class of orthogonal polynomials*, by Mr. Alexander Tartler. (Abstract No. 37–9–331.)

12. *Note on analytic curves everywhere dense in the plane*, by Dr. Wladimir Seidel (National Research Fellow). (Abstract No. 37–11–341–*t*.)

13. *On a certain line locus associated with the lines of two distinct projectively related planes in euclidean space*, by Dr. Lulu Hofmann. (Abstract No. 37–11–342–*t*.)

14. *Normal division algebras of order sixteen over an algebraic field*, by Professor A. A. Albert. (Abstract No. 37–9–329–*t*.)

15. *A third-order irregular boundary value problem and the associated expansion*, by Professor L. E. Ward. (Abstract No. 37–11–343–*t*.)

16. *Affine geometries which are metric*, by Mr. C. A. Lovell. (Abstract No. 37–11–344–*t*.)

17. *The determination of the canonical pencils by means of the linear complexes osculating the asymptotic curves of a surface*, by Professor E. D. Meacham. (Abstract No. 37–11–345–*t*.)

18. *The class of surfaces whose osculating ruled surfaces belong to linear complexes*, by Professor E. D. Meacham. (Abstract No. 37–11–346–*t*.)

19. *Certain tetrahedrons associated with a point of a non-ruled surface*, by Professor E. D. Meacham. (Abstract No. 37–11–347–*t*.)

20. *On the overconvergence of certain sequences of rational functions of best approximation*, by Professor J. L. Walsh. (Abstract No. 37–11–355–*t*.)

21. *Properties of group manifolds*, by Professor P. A. Smith. (Abstract No. 37–11–356–*t*.)

22. *Proof of the analyticity of the surface of a rotating fluid in equilibrium*, by Dr. Eberhard Hopf. (Abstract No. 37–11–357.)

23. *On proposition* *4·78 *of Principia Mathematica*, by Professor B. A. Bernstein. (Abstract No. 37–11–358.)

24. *Seven lines in the real projective plane*, by Professor H. S. White. (Abstract No. 37–9–332.)

25. *Hexagonal systems of seven lines in a plane*, by Professor Louise D. Cummings. (Abstract No. 37–11–359.)

26. *A curiosity in complex geometry*, by Professor Edward Kasner. (Abstract No. 37–11–348.)

27. *On the compactness of the space* L_p, by Professor J. D. Tamarkin. (Abstract No. 37–11–349.)

28. *On the derivatives of potentials on the boundary*, by Professor O. D. Kellogg and Miss Mildred Sullivan. (Abstract No. 37–11–350–*t*.)

29. *Formal theory of linear differential equations*, by Professor Oystein Ore. (Abstract No. 37–11–351.)

30. *Concerning sequences of homeomorphisms*, by Professor H. M. Gehman. (Abstract No. 37–11–352.)

31. *Acyclic, bounded, continuous curves which are homogeneous except for a zero-dimensional set having a non-dense perfect* wth *derived set*, by Dr. T. C. Benton. (Abstract No. 37–11–353.)

32. *A characterization of the closed 2-cell*, by Mr. Hassler Whitney. (Abstract No. 37–11–354.)

R. G. D. RICHARDSON,
Secretary

THE MINNEAPOLIS COLLOQUIUM

The lectures by Professor Marston Morse were concerned with the new theory of the calculus of variations in the large developed by him within the last few years. The intimate relations existing between the calculus of variations and many other branches of modern mathematics were strikingly illustrated by these lectures. For example, combinatorial analysis situs and the theory of analytic functions of several variables form an essential part of the foundations for the new theory, which in turn adds to our knowledge about differential equations and boundary value problems, about dynamics, about "differential geometry in the large." Morse has also initiated a new chapter in analysis situs by studying the connectivity numbers of manifolds in function space associated with calculus of variations problems. Here there may occur infinite connectivity numbers, and infinitely many of the connectivity numbers may be non-zero.

The first lecture was introductory, giving a preliminary outline of methods and results. The nature of the results was indicated by means of simple examples. Then the theory associated with the type number or index of an extremal under general end conditions was developed. This theory includes separation, comparison, and oscillation theorems for systems of self-adjoint linear differential equations of the second order. The index of an extremal is defined by means of the characteristic numbers for an associated boundary value problem, and is shown to be equal to the negative type number of an associated quadratic form. The third lecture was devoted to the inequality relations between the numbers of different types of critical points of functions and the connectivity numbers of the manifolds on which they are defined. The fourth and final lecture took up the connectivities in function space associated with a calculus of variations problem and the inequality relations existing between the numbers of extremals of different types and the connectivities. These inequalities yield some extraordinary existence theorems on the number of extremals satisfying general end conditions. A number of interesting examples illustrated the results.

The lectures were extremely stimulating and instructive. It is greatly to be hoped that the details of the theory outlined in them will soon appear in book form.

There follows the preliminary outline of the lectures prepared by Professor Morse.

"These lectures will be primarily on a theory of the calculus of variations in the large. Such a theory may be regarded as concerned with the most general boundary value problem involving extremals. It requires, I, a preliminary characterization of the solutions in the small, II, a topological study of the critical points of functions and, III, a combination of these results in the form of a general theory. The following topics will be treated.

"I. The Characterization of Solutions in the Small.
- a. *The Lagrange problem under general end conditions.*
- b. *The associated linear boundary-value problems.*
  1. The algebraic properties of self-adjoint systems.
  2. Separation, comparison, and oscillation theorems.
  3. Sturmian conditions as focal point conditions.
  4. Interrelations of conjugate points, focal points, and characteristic roots.
  5. Periodic solutions.
- c. *The geometric invariance of characteristics of solutions.* (Using tensor analysis.)
- d. *The semi-topological invariance of characteristics of solutions.* (Using deformations.)

"II. The Critical Points of Functions.
- a. *The type numbers of critical sets.*
  1. The non-degenerate case.
  2. The analytic case.
  3. The general case.
  4. The invariance of the type numbers.
- b. *Relations between the type numbers and the Betti numbers.*
- c. *The completeness of the relations.*
- d. *Applications to differential topology, potential theory, and dynamics.*

"III. The Calculus of Variations in the Large.
- a. *One end point fixed. Ordinary connectivities.*
- b. *Two end points fixed. Two point connectivities.*
- c. *End points variable on two manifolds. Product connectivities.*
- d. *Periodic extremals. Circular connectivities.*
- e. *A method of topological continuation.*
- f. *The Poincaré Theorem. Birkhoff's results in n dimensions.*
- g. *General formulations and topological analogies.*

"The principal results to be treated have been presented in the literature only in part. The reader may refer to the recent papers by the author in the Transactions, to the Mathematische Annalen, volume 99, and to abstracts in the Proceedings of the National Academy of Sciences. Here will be found references to papers by Birkhoff, Brown, Bliss, Bolza, Richardson, Poincaré, Lusternik and Schnirelmann, Alexander, Lefschetz, Veblen, and others."

L. M. Graves

EDGE ON THEORY OF RULED SURFACES

The Theory of Ruled Surfaces. By W. L. Edge. Cambridge, University Press, 1931. ix+324 pages.

The appearance of the present volume, written by a competent author and published by a recognized house, furnishes further evidence of the general re-awakening of interest in algebraic geometry. The purpose is to describe the algebraic ruled surfaces of orders three to six, by means of the configuration of the double curves and of the developables of bitangent planes.

Only a limited knowledge of projective and analytic geometry is presupposed; a general introduction to the theory of correspondences, geometry on an algebraic curve, and the general properties of ruled surfaces is provided in the Introduction. The plan there is to state the essential theorems, frequently with illustrations, then to give either the reference to its source or to its demonstration in a well known book. The proofs in this preparatory chapter are not included. Two methods of procedure are followed; either one would be sufficient to accomplish the purpose, but when one becomes difficult to apply, the other may suggest new phases of attack. This double derivation provides a comprehensive check on the accuracy of the result.

The first method consists in mapping the lines of ordinary space S_3 on the points of a general quadric manifold M_1^2 of four dimensions and order two in S_5, as proposed by Klein. Ruled surfaces of S_3 are then represented by curves of the same order and same genus on M. The latter are classified according to the dimensionality of the space in which the curve lies; then, if in S_4, according to the number of their intersections with the generating planes of the two systems on M; then, within each category, by the number of actual multiple points. Thus, if C lies in the section of M by a general S_4, the ruled surface belongs to a non-special linear complex, and is properly self dual, that is, each generator is transformed into itself under polarity as to this complex. If the S_4 is tangent to M, the polarity does not exist, and the ruled surface has a rectilinear directrix, which may or may not also be a generator, according as the curve does or does not pass through the point of contact.

No sufficient criteria have been developed to account for all particular forms of the curve which give rise to ruled surfaces having tacnodal or oscnodal multiple curves. With the exception of those surfaces contained in a linear congruence with coincident directrices, no attempt is made to include such cases; indeed their existence is not even mentioned.

The other method is the projection of a normal ruled surface from a higher space, as given by Segre. This is entirely practicable for rational ruled surfaces and for others of low orders, but becomes very cumbersome for higher genera. In the extreme cases the determination of the normal surface reduces to the very problem it is proposed to solve. The methods of projection into S_3 suggest various correspondences by which the surfaces may be generated. Frequently these are not the easiest nor the most rational methods of generation. A chapter on developables includes the proof that all developables of orders less than 8 are

rational. The references to existing literature are much fuller in this chapter than in the others. The curves on M are images of developables when their tangent lines also lie on M.

A chapter on corrections and additions to the earlier types develops the suggestions for finding new ones that are brought out by the chapter on developables. Then follows a table of all the surfaces found, arranged as to order, then genus, then as to multiple directrices which may or may not be also generators, the main basis of classification, within each genus, being the configuration of double curves and of bitangent developables. Finally, a short discussion of interesections of curves on a ruled surface is added.

The book is admirably written, and faultlessly printed. The symbolism is consistent and very helpful. The only typographical error noticed is in the footnote of page 29. The year should be (1895) instead of (1865). The book is not provided with an index.

This excellent treatise contributes a real advance to the theory of ruled surfaces. Although both methods employed were known before, they have been here systematically developed and compared with each other to such an extent as to justify amply the publication of the book.

Two claims (see preface and page 139) made by the author need some explanation. The enumeration is said to be complete, and "so far as is known, no serious attempt has been made to solve this problem before." Since the references to various theorems are extensive, the following should be included: The first algebraic proof that a necessary and sufficient condition that a ruled surface be properly self dual is that it belong to a non special linear complex was given by C. H. Sisam (this Bulletin, vol. 10 (1903–04), pp. 440–441). The same author proved that every rational quintic ruled surface has three coplanar generators (this Bulletin, vol. 10 (1903–04), pp. 32–34). But the fundamental theorem concerning the number of component double curves, the genus of each and the number of its intersections with a generator furnishes a complete criterion for classification so far as rational surfaces is concerned (Sisam, American Journal of Mathematics, vol. 28 (1906), pp. 43–46). A number of omissions would have been noticed by use of this theorem. As to quartics, the comprehensive enumeration of Holgate (American Journal of Mathematics, vol. 15 (1893), pp. 344–386; vol. 22 (1900), pp. 27–30) should be cited. As to quintics, an enumeration including all those in the present volume is in this Bulletin (vol. 8 (1902), pp. 293–296). More details concerning those types having three double conics is found in this Bulletin (vol. 9 (1903), pp. 236–242). Two of the double conics may be replaced by a tacnodal one, or all three by an oscnodal (this Bulletin, vol. 11 (1905), pp. 182–186).

As to sextics, the list of K. Fink (Tübingen doctor dissertation, 1887) and that of J. Bergstedt (Lund dissertation, 1886) should have been mentioned, although neither would seriously affect Mr. Edge's statement. But the essay of A. Wiman, Klassifikation af regelytorna af sjette graden, Lund dissertation, 1892, contains 118 forms, including all mentioned in the present volume, and a considerable number of others. The basis of classification is different, but a wealth of detail concerning double curves and bitangent developables is given. It is synthetic, and maps the generators of the surface on the points of a curve, in ordinary space. In a series of papers in the American Journal of Mathematics,

vol. 25 (1902), pp. 59–84, 85–96, 261–268; vol. 27 (1905), pp. 77–102, 173–188, the present reviewer established the existence and derived the equations of about two hundred sextic ruled surfaces, including all those in the book under review, and about eighty others which under Mr. Edge's basis of classification should be counted as distinct types. No criterion was found, to establish the completeness; thus far, no other forms have been published.

Of the specific omissions, the following have distinct double curves:

two double cubics and two double conics	$p=0$;
four double conics+two double generators	$p=0$;
double cubic and three double conics	$p=1$;
double quartic, two double conics and a double generator	$p=1$;
double quartic, double conic, double directrix and two double generators	$p=1$.

Then there are fourteen other forms without rectilinear directrices, having tacnodal or oscnodal curves. Thus, of the four double conics, two may approach coincidence, forming a tacnodal conic, or three may approach coincidence, forming an oscnodal conic. Of the large number of omissions of those having a directrix line which may or may not be a generator, a frequent sample is that caused by a compound involution.

For ruled surfaces having a directrix line, the reviewer is inclined to feel that the methods of Wiman and of Sisam, American Journal of Mathematics (vol. 29 (1907), pp. 48–100), are at least as powerful and as comprehensive as those developed by the author. Since all these papers are featured in the Encyklopädie (III C 8; Art. 52), the author should at least have mentioned them. There, if the reviewer has established his reasons for protesting the content of one paragraph, he now wishes to emphasize that Mr. Edge has produced an excellent book that will be of very great value in the study of various branches of algebraic geometry.

VIRGIL SNYDER

HITHERTO UNPUBLISHED TREATISE OF STEINER

Allgemeine Theorie über das Berühren und Schneiden der Kreise und der Kugeln worunter eine grosse Anzahl neuer Untersuchungen und Sätze vorkommen in einem systematischen Entwicklungsgange dargestellt. By Jakob Steiner. Edited by Rud. Fueter and F. Gonseth. Zürich and Leipzig, Orell Füssli, 1931. 8vo, xviii+345 pp. Price in marks: paper, 10.80; cloth 12.80.

This is an original manuscript written by Steiner more than a century ago. It now appears in print for the first time. It is published under the auspices of the Swiss Naturalist Society with the asistance of the Escher-Abegg Foundation for Scientific Research at the University of Zürich. It appears as volume five in a series of publications of the Swiss Mathematical Society.

Jakob Steiner (1796–1863), the great Swiss geometer, wrote this treatise on the circle and the sphere probably during the years 1823–26 while a private teacher in Berlin. The manuscript consists of 360 carefully written pages with title, book and chapter headings, evidently all prepared for immediate publication. For some reason, however, it was never published. The editors suggest

that it may have been intended for the geometrical encyclopaedia planned at that time by Steiner, but never realized.

An earlier manuscript, probably written in 1825, whose table of concents is complete, was to contain three books, the first treating the circle, the second, the sphere, and the third, circles on the sphere. Of this work, only the first two chapters and a part of the third in the first book were written up. This manuscript was evidently superseded by the one now published, which, however, does not contain a discussion of circles on the sphere.

Steiner dated many of his manuscripts, but only one date, Oct. 1, 1825 at the beginning of book three, occurs in the completed manuscript. This date and certain references by Steiner in published memoirs indicate that this work was finished in 1826.

The question naturally arises: Why was this important treatise, painstakingly prepared for publication by Steiner himself, not included in Steiner's Gesammelte Werke (1881–82), edited by Weierstrass with the assistance of Geiser?

In the preface of the second volume of the Gesammelte Werke, Weierstrass writes: " . . . The expectation shared by many that this volume would contain a series of interesting, hitherto unpublished articles from the Steiner manuscript legacy (Nachlass) rests on a false impression of the contents of this legacy. Professor Geiser of Zürich has had the goodness to inform me that it consists chiefly of works in preparation and various drafts of a number of memoirs already published and that they would require thorough revision to be of use . . . ."

What happened to most of Steiner's manuscript legacy is told by Bützberger:* "Long after the death of Steiner, this manuscript was found on the floor of the Library of the Naturalist Society of Berne where it had been left in a box, disarranged, forgotten and in part already decomposed. Professor Graf found it there and turned it over to the author to put it in order and turn it to good account."

The title given by Bützberger a few lines further on is evidently that of the older manuscript which was never finished. He states that numerous other Steiner manuscripts were found in this box. He does not give the date of this discovery, but it had evidently happened only a short time before—probably at least thirty years after the death of Steiner and twelve years after the publication of his collected works.

Bützberger had ten volumes of Steiner's lecture notes, letters, etc. bound and placed in the Library of the University of Berne. He kept the more important manuscripts and published results from some of them later,† but died before he had completed the work of editing the Steiner manuscripts.

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* F. Bützberger, *Zum 100 Geburtstage Jakob Steiners*, Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, vol. 27 (1896), p. 162, beginning with line 24.

† F. Bützberger, *Ueber Bizentrische Polygone, Steinersche Kreis- und Kugelreihen und die Erfindung der Inversion*, Leipzig, Teubner,1913. See also review by Emch, this Bulletin, vol. 20 (1913–14), pp. 412–415.

In the summer of 1928, Arnold Emch* examined the completed manuscript on the circle and sphere, then in possession of Mrs. Bützberger at Zürich. Very probably its present appearance is due, at least in part, to his efforts to interest the Swiss Mathematical Society in its publication.

The treatise is divided into four books, each containing two chapters further subdivided into sections numbered continuously through the entire volume.

The first book deals with centers, lines of similitude and planes of similitude of circles and of spheres; the second with powers and loci of equal powers of circles and of spheres; the third with common powers of circles and spheres and the fourth with systems of circles and systems of spheres that intersect at given constant angles. In each of the books, the first chapter deals with certain properties of circles and the second with the corresponding properties of spheres. In each chapter, theorems are stated and proved, followed by problems and figures illustrating these properties.

The treatment in the fourth book is very general and complete. Here are found the most important contributions of the work—properties rediscovered by mathematicians many years later and often in a more restricted form, for example, orthogonal systems of circles and spheres instead of systems intersecting at any given angle. Here Steiner also develops the properties of surfaces that are envelopes of systems of spheres. In particular, the surface discussed on pages 261–267 is the Dupin cyclide.

One of the great advances in method instituted by Steiner was to treat contact as a special case of intersection; another was a happy system of notation that reduced complicated problems to simple form. Both of these methods are made use of freely in the present volume.

Highly important as this treatise is in content and method, it is fully as valuable as a complete and perfected production of the great geometer, revealing the clearness, simplicity and rigor of his style. The editors have taken great care to preserve this work as a classic and have made no attempt to introduce changes where, in the light of our present knowledge, improvements could be made. Annotations have been inserted by the editors only where considered absolutely necessary, "small additions that Steiner himself would have made for the press in his lifetime," and these are enclosed in parentheses.

Some of the figures, merely sketched by Steiner, have been redrawn more carefully, but the only one completely new is Fig. 60, the large diagram on the extension of the last page. Even in this figure, the basic circles and lines were drawn by Steiner (Fig. 48). The diagram was completed by J. Züllig from Steiner's description of it.

The excellent typography, paper and binding of the volume are in keeping with the worthy treatise to which they lend added years and a wide sphere of usefulness. The editors, the publishers and all directly or indirectly concerned in the preservation and publication of this manuscript richly deserve the deepest gratitude of every mathematician. In reality, after a half century, another volume has just been added to Steiner's Gesammelte Werke.

T. R. HOLLCROFT

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* A. Emch, *Unpublished Steiner manuscripts*, American Mathematical Monthly, vol. 36 (1929), pp. 273–275.

WEATHERBURN'S SECOND VOLUME

Differential Geometry of Three Dimensions. Volume II. By C. E. Weatherburn. Cambridge University Press, 1930. xii+239 pp.

The first volume of this work (see this Bulletin, vol. 34 (1928), pp. 785–786) covered the fundamentals of metric differential geometry of curves and surfaces. The present volume, though containing certain classical material supplementing that of the first volume, is primarily devoted to a consequential exposition of the author's published contributions to the subject.

The treatment in both volumes is in terms of vectors. But, whereas the first volume employs, except in the last chapter, on differential invariants, only the algebra of vectors, the second volume uses also, and to a great extent, the differential and integral calculus of vectors. Moreover, dyadics are introduced in the middle of the volume and are used to good effect throughout the later chapters.

Chapters 2, 3, and 8 contain the author's work on families of curves on a surface. In Chapters 5 and 6 the calculus of vectors referred to curvilinear coordinates in space is developed and applied to families of surfaces. Chapters 10, 11, and 12 have to do with transformations, small deformations, and applicability of surfaces. In Chapter 13 are found the author's contributions to the theory of curvilinear congruences, and in Chapter 9 those bearing on the parallelism of Levi-Civita.

In treating the family of curves $\phi(u, v)=\text{const.}$ on a surface, the author makes extensive use of the function $\psi=1/(\Delta_1\phi)^{1/2}$. Since the distance between the curve $\phi=\phi_0$ and the curve $\phi=\phi_0+\Delta\phi$, measured along the orthogonal trajectories of the curves $\phi=\text{const.}$, is approximately $\psi\Delta\phi$, he calls ψ the distance function for the given family of curves and the curves $\psi=\text{const.}$ the lines of equidistance for this family. Similarly, in treating a family of surfaces in space $\phi(u, v, w)=\text{const.}$, he introduces the distance function $\psi=1/(\Delta_1\phi)^{1/2}$, the surfaces of equidistance $\psi=\text{const.}$, and the lines of equidistance $\psi=\text{const.}$, $\phi=\text{const.}$

The distance function enables the author to simplify greatly the formal work. His skilful use of it in connection with the study of ruled surfaces in Chapter 4 is particularly to be commended. There is, however, an important point in connection with it which he fails to mention. Though the function itself is invariantly connected with the given function ϕ, the lines of equidistance for a given family of curves on a surface and the surfaces of equidistance for a given family of surfaces are not fixed respectively by the family of curves and the family of surfaces; they vary with the choice of the function ϕ employed to define the given family. On the other hand, the lines of equidistance for a family of surfaces are uniquely determined by the family itself.

In applying dyadics to a family of curves on a surface, the author introduces what he calls the tendency, the moment, and the swerve of the family in a given direction. If $\boldsymbol{t}$ is the unit vector tangent to the curve of the family passing through a point P and $\boldsymbol{a}$ is the unit vector in the given direction at P, these

quantities are respectively $\mathbf{a}\cdot\nabla\mathbf{t}\cdot\mathbf{a}$, $\mathbf{a}\cdot(\nabla\mathbf{t}\times\mathbf{t})\cdot\mathbf{a}$, and $\mathbf{a}\cdot(\nabla\mathbf{t}\times\mathbf{n})\cdot\mathbf{a}$, where $\mathbf{n}$ is the unit vector normal to the surface at P. More familiar expressions for, say, the first and the third are $\sin\omega(\gamma\cos\omega+\gamma'\sin\omega)$ and $\cos\omega\,(\gamma\cos\omega+\gamma'\sin\omega)$, where ω is the angle from $\mathbf{t}$ to $\mathbf{a}$ and γ and γ' are respectively the geodesic curvatures of the given curves and their orthogonal trajectories.

It would appear that these concepts were motivated more by the vector analysis than by the geometry of the situation. On the other hand, similar concepts are employed, in the study of curvilinear congruences, to yield useful and striking geometric results.

Levi-Civita's concept of parallel displacement is set forth in excellent fashion except for the omission of the definition, in terms of it, of geodesic curvature. An interesting application consists in the theorem to the effect that a system of curves on a surface clothes the surface in the sense of Tchebychef if and only if the unit vectors tangent to the curves of each family are parallel with respect to the curves of the other family.

In the map of a surface S on a surface S' parameters u, v common to the two surfaces are chosen so that corresponding points have the same coordinates (u, v). The discussion is restricted to maps for which the components dx', dy', dz' of the vector element $d\mathbf{r}'$ on S' are linear homogeneous functions of the components dx, dy, dz of the corresponding vector element $d\mathbf{r}$ on S, or, what is the same thing, to maps for which $d\mathbf{r}'=\Phi\cdot d\mathbf{r}$, where Φ is a dyadic depending on u, v. The map is thus made to depend on a dyadic and the study of it is in terms of this dyadic.

The small deformations of a surface discussed by the author are perfectly general; so-called infinitesimal deformations, which preserve the linear element except for terms of the second and higher orders, are considered briefly as a special case. The treatment of applicability of surfaces is along classical lines.

The greater part of the references in the book are to the author's published papers. While this is quite proper, inasmuch as the author's contributions are largely novel, there are regrettable oversights of original sources in the case of a few well known theorems.

The tendency of the vector analysis to override the geometry is greater in this than in the previous volume. Only the author's strong interest in geometry and keen geometrical insight save the book from becoming too formal. As an extension of the first volume and an exposition of the author's widely published researches, it should make a strong appeal both to the student and to the specialist.

W. C. GRAUSTEIN

TWO BOOKS ON FOUNDATIONS OF GEOMETRY

Grundlagen der Geometrie. By Gerhard Hessenberg. Edited by W. Schwan. Berlin, de Gruyter (Göschens Lehrbücherei, 1. Gruppe, Band 17), 1930. 143 pp., 77 figs.

Vorlesungen über Grundlagen der Geometrie. By Kurt Reidemeister. Berlin, Springer (Die Grundlehren der mathematischen Wissenschaften, vol. 32), 1930. x+147 pp., 37 figs.

It is well that the simultaneous publication of two books on the bases of geometry, both by most competent authors, both making noteworthy contributions to the subject, should remind us of the progress which still continues after so many centuries. In writing on this topic, an author chooses between the exclusive use of strict logic, on the one hand, and, on the other, enough of an appeal to the pictorial sense to illuminate the subject and make it more accessible to readers with but a moderate power of abstraction. Reidemeister's book is of the first type; while that which was begun by Hessenberg and, after his death, completed by Schwan, is rather of the other kind.

The latter book, to be sure, almost completely satisfies a logician's demands, but there are a few diverting gaps in the argument—probably the result of divided authorship. Thus (p. 33), when the nature of lines and their segments has been carefully discussed, parallelism and direction are introduced with no foundation, no mention that there is such a thing as a plane. On page 77 we find the first use of the word "motion,"—"Let a plane E, which coincides with A, be so moved . . . ," and there is no hint of the concept of motion or its relation to congruence.

Distinctly good is the discussion of the logical relations between the Pascal and Desargues theorems, axioms of continuity (particularly that of Archimedes), order, incidence and parallelism, and the fundamental theorems of plane and solid projective geometry. There is, of course, Hessenberg's proof that Desargues' Theorem is a consequence of Pascal's. Important "artificial" geometries illustrate certain independences. Thus we have Moulton's non-desarguean plane geometry and various non-archimedean geometries. The latter group of examples shows how the axioms of incidence, order and parallelism are insufficient to prove Pascal's Theorem, and how the axioms named, plus this theorem, are still not enough to establish the axiom of Archimedes. Since Pascal's Theorem, or its corollary, Desargues', is sufficient for the transition from two dimensions to more, the problem is proposed of discovering what axiom, weaker than the archimedean, will suffice to establish this transition without the use of congruence.

After a short, clear initial chapter on the concepts of equality, order, and continuity, that of congruence leads up to a discussion of numerical measurement. Hessenberg treats measurement of segments, Schwan continues with that of vectors—with the gap in definitions already noted. In the measurement of angles, Schwan distinguishes between the angles themselves (figures consisting of two half-lines and, in measure, never exceeding π) and angular fields

(portions of the plane swept over in possible motion from one side to the other). Angles, like complex numbers of absolute value 1, are combined by "multiplication," fields by "addition." There is some convenience, from the point of view of elementary geometry, in having a pair of half-lines furnish but one angle; yet there is awkwardness also, and it would seem simpler to be content with "equality" and "congruence" of unrestricted angles. We can hardly agree that "the multiplication of two obtuse angles yields an acute one." Finally, Hessenberg treats adequately of the measurement of plane areas (with emphasis on Hilbert's work on non-archimedean geometry) and of solids.

Chapter 3 treats of projective geometry in the plane. Besides the emphasis on the logical relation between the basic theorems and axioms, we find the clever schematization of the Pascal configuration very pleasing. It is only at the close, by the introduction of ideal points, that the plane truly becomes projective; the requirement of continuity is avoided, as is frequently the case in this book, the device here being the use of reflections and motions in the definition of ideal points.

A noteworthy feature of the treatment of spatial projective geometry is that "artificial" geometry of Hessenberg which interprets each point in space as a triangle in a fixed plane whose sides have given directions. This device, suggested by descriptive geometry, elegantly illustrates the desarguean transition from plane to solid geometry. Coordinates on a line are set up in a method whose distinctive feature is the use of a particular line at infinity.

In both the books here reviewed there is interesting and enlightening comment on relations between geometries on divers foundations, and on the contact of geometry with philosophy on the one hand, and with the physical world on the other. A paragraph by Hessenberg, which illustrates this, would serve even better to elucidate Reidemeister's work than his own. "Only that part of geometry should be called 'coordinate geometry' which numerically fixes objects that can be geometrically visualized. We have the purest form of coordinate geometry when these numbers are not taken from the sphere of arithmetic, but derived from geometric relations. 'Analytic geometry' is something quite different. In its purest form it avoids reference to visualizable reality; in place of that, it builds an artificially abstract geometry with the numbers of arithmetic."

Except when setting up artificial geometries, Hessenberg gives us coordinate geometry. Reidemeister, in his remarkably well planned work, has a section devoted to each type. The care with which the design was made is indicated, not only by the contents, but also by the omissions. Thus we miss, of the topics treated in the other book, angles, superficial and spatial measurement; and we should probably never hear of a third dimension, were it not for its connection with Pascal's theorem. Points and lines in a plane are all,—but it is often a more general plane than Euclid could have imagined. There is no mention of the classical non-euclidean geometries.

In the introduction,—abstract, simple, general,—the idea of congruence is basic; congruence is the equivalence of elements in any universe under any group of transformations. A second important concept is that of the reference set (Bezugsmenge),—a set of elements (not exhausting the class under consideration) such that every non-identical transformation of the group considered displaces at least one element. Clearly the addition of any number of elements to a

reference set yields either the entire class of elements or else another reference set—a fact which might well have been mentioned. Usually Reidemeister speaks only of reference sets of minimum size. Thus, if the elements are the points of a plane (pairs of real numbers), a reference set for the group of motions consists of two points at a fixed distance, that for the affine group of three linearly independent points, and for the projective group of four points, of which no three are linearly dependent.

Only when each element is a set of real numbers is the extent of the reference set left arbitrary, and that is merely in the first definition of "natural coordinates." A particular reference set $\mathfrak{B}_0$, consisting of m n-ples, is chosen. If another reference set $\mathfrak{B}$, together with an element Z corresponds, by a transformation of the group considered, to the set $\mathfrak{B}_0$ and the element Z_0 (which consists of the numbers $(z_1, z_2, \cdots, z_n)$), then these numbers z_i are the natural coordinates of Z with respect to $\mathfrak{B}$. The values of m and n are not restricted.

The second chapter develops algebras from the axioms of operations and of order, so that analytic geometry may be built on logically founded number systems. Reidemeister considers not only the number systems obeying all the usual axioms of operations—and therefore forming fields—but also, and with greater interest, those in which one or another axiom does not hold. By analogy with Hessenberg's language, we might speak of "artificial algebras." Especial emphasis is laid on non-commutative multiplication, but we have also non-archimedean ordered fields and unilaterally distributive number systems.

There is one part of this chapter which is not quite satisfactory—a section entitled "Real numbers as ordered groups," which leads the way to the arithmetization of vectors in Part II. Its object is to show that the real numbers constitute but a special case of systems satisfying the following requirements:

1. The numbers $a, b, \cdots$ with the combination $a+b$ form an ordered group. (Let us call it $\mathfrak{G}$.)

2. Every number can be uniquely halved.

3. If a and b are two numbers and if, for every rational r, $r>a$ and $r>b$ are either simultaneously true or simultaneously false, then $a=b$. (This is equivalent to Archimedes' axiom.)

Now bisection yields, not all rational numbers, but only those of form $k/2^n$. Accordingly, the assumption that every cut in these dual fractions can correspond to but one real number is not, without further analysis, justified by Axiom 3.

Once it is demanded that to every cut there correspond a number of $\mathfrak{G}$, the numbers of the class are the products of the real numbers and of a. Each automorphism of their group has the form $\xi=\xi\iota$, ι being a real number. Reidemeister then states that the class consists of the real numbers themselves only if $\iota=1$. This is correct, but the true reason seems not to be mentioned. There is no provision for multiplication within $\mathfrak{G}$, and therefore no unit element. Were there a unit and multiplication, then $\mathfrak{G}$ would be isomorphic with the system of real numbers.

Chapter 3 develops affine and projective geometry. The most notable feature is the omission of the axiom of commutative multiplication. Of course the algebra must be developed without determinants, and it is done elegantly. A weakness for the word "umgekehrt" is noted. If the hypothesis of a theorem has five items, of course there are five possible converses. In Chapter 5, Reidemeis-

ter speaks of "the further converses of Theorem 1". What, then, can we make of the last sentence of this theorem? "Let a_1 and a_2, $\bar{a}_1$ and $\bar{a}_2$ be two pairs of parallel vectors; if, then, $a_2 = a_1 a$, $\bar{a}_2 = \bar{a}_1 a$ and $a = \bar{a}$, then the vectors $a_1 - \bar{a}_1$ and $a_2 - \bar{a}_2$ are parallel. The converse is also true, provided a_1 and $\bar{a}_1$ are not parallel."

The second part of the book, embodying results published in the last four years by Reidemeister and others, builds up geometry on non-algebraic axioms. The connection with analytic geometry is then established through geometric entities which correspond to numbers. For this purpose there is a separate axiom equivalent to each axiom on algebraic operations; and every such axiom is a statement of the closure of a particular figure—of the incidence of a point and a line in a configuration. For instance, Pascal's theorem holds if and only if multiplication is commutative. Commutativity of addition is given by the following axiom (2)—in which $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ lines are, respectively, members of three coplanar parallel pencils— "If P_1, P_2; Q_1, R_1; Q_2, R_2 lie on three $\mathfrak{A}$ lines; R_1, R_2; Q_2, P_2; Q_1, P_1 on three $\mathfrak{B}$ lines; Q_1, Q_2; P_2, R_1 on two $\mathfrak{C}$ lines, then R_2, P_1 also lie on a $\mathfrak{C}$ line" (p. 55).

This system of lines, called a 3-web (3-Gewebe) is the frugal equipment with which the first axioms, and a wealth of subsequent developments, are concerned. Incidentally, Reidemeister briefly discusses an unusual type of "improper points." The composition of vectors is indicated by multiplication, homogeneous coordinates replace non-homogeneous ones, and the points on the fundamental triangle are named improper. The three pencils are now determined by the triangle's vertices, and thus the proper points of the 3-web are mapped on the interior of the triangle. (There is similarity to the interior of a triangle inscribed to the absolute conic in hyperbolic geometry, but it is only the straight lines through the vertices in one case which obtain linear equations in the other.)

The addition of a fourth pencil $\mathfrak{D}$, with a proper point as vertex, and of lines obtained from these by translations in the $\mathfrak{A}$ direction, completes the material for a plane. Simple projective operations define equality and proportionality, addition and multiplication, so that each ratio of collinear vectors can be called a number. The noteworthy feature of this development is the geometric basis for number systems which need not obey all the usual operational axioms. If all of these, except perhaps commutativity of multiplication, are satisfied, the plane is affine.

The final chapter is devoted to projective geometry, the transition from affine geometry having again as chief novelty non-commutative multiplication. Thus, the fundamental theorem takes the form of this statement on cross ratios. "If γ and γ^* are two projectively related lines, and U, O, E, V correspond to U^*, O^*, E^*, V^*, then there is such a number v_0 that

$$v(U^*, O^*, E^*, V^*) \equiv v_0 v(U, O, E, V) v_0^{-1}."$$

Pascal's theorem then brings us to the usual geometry. The book closes with a clear discussion of the relation between its two parts, of the need for analytic geometry, and with a statement of problems as yet unsolved.

A few changes would make the work easier reading. If an automorphism were always given that name, there would be a gain. In the statement of un-

usual algebras there should be greater care. Thus, on page 38, line 4, ρ^3 should be replaced by ρ. On page 41, the sum of $a=(\alpha_1, \alpha_2)$ and $b=(\beta_1, \beta_2)$ is of course not $(\alpha_1+\alpha_1, \beta_1+\beta_2)$, but $(\alpha_1+\alpha_2, \beta_1+\beta_2)$.

Excessive brevity, especially toward the end, is hard on the reader. Equality of $\mathfrak{A}$ vectors in an $\mathfrak{ABC}$ web (Vektorgleichheit) and in an $\mathfrak{ABD}$ web (Masszahlengleichheit) do not mean the same thing, yet they are both called equality; this is especially confusing when, as on page 117, all four pencils are named. And, finally, it is to be hoped that the author will supply the next edition with an index and with an appendix containing all the axioms, to which reference must frequently be made.

Both books are excellently printed. Springer draws attention to his clear, neat page by a footnote on page 130 in contrasting type.

A good geometrical library will certainly wish to possess both works, the more so since they so surprisingly avoid duplication of subject matter. If a reader studies first the Hessenberg work, with its appeal to the pictorial sense, and then the sterner, very stimulating book of Reidemeister, he can gain greatly in his knowledge of the logic of geometry.

E. S. ALLEN

FOUR FRENCH BOOKS ON HYDRODYNAMICS

(1). *Leçons sur la Hydrodynamique.* By H. Villat. Paris, Gauthier-Villars, 1929. iii+296 pp.

(2). *Leçons sur la Théorie des Tourbillons.* By H. Villat. Paris, Gauthier-Villars, 1930. ii+300 pp.

(3). *Mécanique des Fluides.* By H. Villat. Paris, Gauthier-Villars, 1930. vii+175 pp.

(4). *Leçons sur la Résistance des Fluides non Visqueux.* By P. Painlevé. Première partie rédigée par A. Metral. Paris, Gauthier-Villars, 1930. iv+183 pp.

Within the last decade or two the interest in research in hydrodynamics has spread widely throughout all European countries, almost without exception, and very important progress is being achieved in various branches of theoretical and applied hydrodynamics. It is the more striking and disappointing that America has practically no share in this recent progress of hydrodynamics, at least as far as theory is concerned. The amount of papers on theoretical hydrodynamics published in this country is practically negligible. This is deplorable not only because of the high intrinsic interest and practical value of hydrodynamics, but also because in hydrodynamics we have an excellent field of application and means of sharpening of various mathematical tools. Suffice it to say that the recent progress of hydrodynamics mentioned above is intimately related to the modern developments of the theory of functions of a complex variable, conformal mapping, potential theory, integral and integro-differential equations and the like, being in many cases not only the after-effect but even the cause of the development. The last but not the least important aspect of hydrodynamics is the pedagogical one: there is perhaps no better way of illustrating abstract mathematical theories and of developing in students a critical ability in applying and interpreting mathematical results. It is to be hoped that the recent books published in Germany (by Oseen and Lichtenstein)

and in France (books under review) will awaken interest in hydrodynamics and will place hydrodynamical research in this country on a fitting level.

The modern progress of hydrodynamics is well reflected in the books under review, at least in those by H. Villat. They are devoted to the hydrodynamics of an incompressible (perfect or viscous) fluid, of which the irrotational motion is discussed in (1), (2) being concerned primarily with the theory of vortices.

In Chapters 1–4 of (1) we have an excellent exposition of some indispensable results of the theory of functions of a complex variable and conformal mapping. We note particularly the elegant introduction of elliptic functions in connection with the solution of Dirichlet's problem for a circular ring between two concentric circles, and explicit formulas for the conformal mapping of a simply-connected domain on the unit circle (due to Cisotti); these formulas are applied to a derivation of the classical Schwarz-Christoffel formulas for the mapping of polygons. After recalling the fundamental equations of motion of perfect fluids (Chapter 5), we pass on to a detailed discussion of the theory of liquid jets (originated by Helmholtz and Kirchhoff), Chapters 6–8. Investigations of several writers, those of the author himself inclusive, are discussed here in a very clear fashion. Various difficulties in the application of abstract results are pointed out and some remarkable cases of the multiplicity of physically admissible solutions are treated. The rest of the book is devoted to problems of motion of a viscous (incompressible) fluid, mainly to that of the motion of a rigid body in such a fluid. The author succeeds completely in the difficult task of giving a rapid but clear and accessible exposition of the complicated theory developed by Oseen and his pupils, particularly by Zeilon. The most important problem of this theory is the discussion of liquids with small viscosity and the investigation of the limiting solutions obtained when the coefficient of viscosity tends to zero. This is done in Chapters 14–16, on the basis of Oseen's integral equations (Chapter 13), after the equations of a viscous fluid have been derived (Chapter 9) and certain important properties of fundamental solutions of the heat equation (Chapter 11) and of Newtonian potentials (tangential derivatives of potentials of a simple spread, Chapter 12) have been established. An important result of the theory of Oseen—that the motion of a perfect fluid is *not* the limiting case of the motion of a viscous fluid whose coefficient of viscosity tends to zero—had already been observed in a very special case by Boussinesq. The analysis of this case (Chapter 10) helps the reader considerably in the understanding of the much more general theory of Oseen. The concluding Chapters 17–20 are given to the treatment of special cases, in two and three dimensions, on the basis of papers by Zeilon. An elegant application of properties of Legendre's functions is pointed out in Chapter 19.

Book (2), as follows from its title, treats of the theory of vortices and turbulent motion. In Chapters 1–2 general equations are derived and formulas expressing the velocities in terms of the vortices are given. Chapters 3–6 deal with problems arising in the theory of plane motion of rectilinear vortex lines: canonical equations, stability questions, vortices of Kármán-Bénard and their various extensions, computation of the resistance of the liquid due to turbulent motion. Interesting remarks concerning applications of conformal mapping to the problems of vortex motion in two dimensions are found in Chapter 7. Chapter 8 treats of some special problems of vortices of finite dimensions

(elliptic vortex of Kirchhoff and spherical vortex of Hill), while vortex rings (of infinitely small section) are discussed in Chapter 9. Chapter 10 recalls some fundamental facts of the theory of propagation of discontinuities, due mainly to Hadamard. Chapters 11–13 are devoted to a brief exposition of the fundamental existence theorem for the equations of hydrodynamics. It is rather unfortunate that the method used here is ascribed to Lichtenstein alone, whose name only is mentioned in the book; no reference is made to the important work of Günther, which deserves as much credit as that of Lichtenstein. The concluding Chapter 14 deals with some special cases of vortices in a viscous fluid. It is seen from this rapid survey that several important problems of hydrodynamics, both classical and modern, are not touched upon in the two excellent monographs of H. Villat, for example, figures of equilibrium of celestial bodies, wave-motion, hydrodynamics of compressible fluids and of those with radiation of energy, including the dynamics of atmosphere, statistical methods. While this appears to be quite natural and wise, on the whole, it is the reviewer's opinion that the important investigations of Friedmann concerning the kinematics of vortices of fluids that do not obey Helmholtz's laws should be treated in a more detailed way than is actually done in Chapter 1. Problems of turbulent motion also might have been treated in more detail, and a brief exposition of statistical methods would not be out of place in a treatise on modern hydrodynamics: it is by these methods that, in all likelihood, a true theory of resistance of fluids will be found, and mysteries of the dynamics of atmosphere will be explained.

Book (3) is of a more elementary and less inclusive character than the preceding two, with which it overlaps in greater part. As interesting additions we should mention, however, an excellent account of the Kutta-Joukowsky Theorem with its applications and discussion of cases of exception, and an exposition of Prandtl's theory of carrying surfaces. All three books by H. Villat under review are marked by their clear and suggestive style of exposition. The misprints, although rather frequent, do not spoil appreciably the pleasure of reading these books. We are looking forward to the promised continuation of these monographs, *Sur la Répresentation Conforme et quelques unes de ses Récentes Applications*, which is now in press.

Book (4) by P. Painlevé is on a level quite different from those by H. Villat, both as to choice of material and exposition. Out of its 183 pages only 88 treat of hydrodynamical subjects at all, the remaining space being given to an outline and mathematical preparations. The most interesting part of the book is the "préambule" (on theoretical hydrodynamics and experimental laws of air resistance), incomplete as it is. The exposition of other chapters is confined to generalities that might be found in any standard treatise on dynamics of fluids, without illustration by concrete examples. The misprints and lapses, corrected in the Errata, as well as uncorrected ones, are abundant. The final judgment about Painlevé's *Leçons* should be reserved until the publication of their second part, the content of which is unknown at present. It is probably there that the author expects to succeed in his efforts to "pousser aussi loin que possible la solution théorique de ce grand problème" (rigorous solution of the problem of a motion of a rigid body in a perfect liquid).

J. D. TAMARKIN

SHORTER NOTICES

A Manual of Greek Mathematics. By Sir Thomas L. Heath. Oxford, Clarendon Press, 1931. xvi+552 pp. Price $5.00.

When Sir Thomas Heath's *History of Greek Mathematics* was published in 1921, it was evident that these two volumes not only summarized the author's studies of special writers but that they filled the gaps necessarily left between these works. At first sight, the purpose of the *Manual of Greek Mathematics* is not so clear, and one is tempted to suppose that it is merely a condensed version of the other work. This, however, is not the case, for there are important differences in purpose and in method of presentation.

The *History* was intended for classical scholars "who might look for light on the interpretation of passages of mathematical content in Greek authors" and for the expert mathematician who might wish to master the method of particular individuals with a view of applying it to other problems. The *Manual* was written for "the general reader who has not lost interest in the studies of his youth." One suspects that the "general reader" in America has little curiosity about the contributions of the Greeks to mathematics. He has studied "geometry" rather than "Euclid" and he presumably has had less classical training than has his fellow in England. Accordingly, this *Manual* is likely to find its readers,—and there should be many of them,—among those students and teachers whose interest in mathematics prompts them to learn more about its development.

Like the *History of Greek Mathematics*, the material in the *Manual* is arranged mainly according to the subject matter, a scheme that shows the greatest of the Greek mathematicians against the background of the earlier development of their respective fields. This similarity in arrangement results in some cases in identity of chapter headings and at times in the repetition of key sentences, but while certain minutiae present in the first are omitted in the second, other matters are given greater explanation. For example, in discussing Pythagorean arithmetic, the author says in the *History* (I, p. 90), "This subject (the irrational) was regarded by the Greeks as belonging to geometry rather than to arithmetic." In the *Manual*, the corresponding section (p. 54) states that "The subject of irrationals in general was for the Greeks a part of geometry rather than arithmetic, and necessarily so, because, for want of notation, an irrational of any sort could only be denoted by a straight line or a combination of lines." Other instances of this kind are to be found in great numbers, each one clarifying a particular point which the readers for whom the *History* was designed were expected to supply for themselves. The volume is thus well adapted to the needs of the beginning student in the history of mathematics, but it will be useful also for the more advanced worker.

In addition, just as the *History* presented materials not at hand when the earlier studies appeared, so the *Manual* contains the results of recent researches especially in the fields of Babylonian and Egyptian mathematics which are treated in the earlier chapters.

VERA SANFORD

Geometry of Four Dimensions. By A. R. Forsyth. Cambridge University Press, 1930. Two Volumes, Vol. I, 468 pp., Vol. II, 520 pp.

In the past few years several books on the geometry of four dimensions have appeared. They have been written, usually, with the aim of presenting the geometry necessary for the understanding of the theory of relativity. The present book has no such end in view. The author wishes simply to present a complete account of that geometry. The question then arises, why four dimensions and not n dimensions. We are told, in the preface, that there are two reasons for this; first, four dimensions is the simplest generalization of three dimensions and by treating these generalizations in minute detail much may be gained; second, there is a traditional importance to four dimensions.

One might naturally suppose that these two volumes would contain all that is known about the geometry of four dimensions, but this is not the case, as one more or less familiar with the subject soon finds out. The general plan of the work is excellent and it is written in the characteristic Forsyth style. The method used is that of Gauss and the notation is that of Cayley. The treatment is wholly analytical. The space is euclidean and in the preface we are told that curvature of a curve measures its departure from a euclidean straight line and that the complete curvature of a surface is a measure of its deviation from an euclidean plane. It seems to me that these statements need considerable interpretation since a fixed plane, for example, is euclidean or not according to the measure used. The only reason I can see for assuming euclidean space to be at the bottom of things is because it is the simplest known.

The work is divided into five main parts:

I. The first part (203 pages) is an account of the geometry of lines, planes and hyperplanes and the relations of one to another. I know of nowhere else that this material can be found in one place. A complete discussion of parallelism and perpendicularity of these elements and of the shortest distance and angles between them is given. A chapter on rotations in four dimensions is also included.

II. Part two (147 pages) is devoted to the theory of curves. The treatment is quite analogous to the theory of curves in ordinary 3-space but we have a new curvature, the second torsion or tilt, which complicates the theory to some extent. A chapter on curves in n-space is also included.

III. Part three (115 pages) is on surfaces in 4-space. While some parts of the theory of surfaces in 4-space are quite analogous to that of surfaces in 3-space there are other parts of it which are quite different. Geodesics are analogous, but properties of ordinary surfaces which depend upon the normal do not generalize quite so easily. Lines of curvature and asymptotic lines are examples. Gaussian curvature is the same; but in four dimensions, mean curvature is a vector. There are other properties which depend upon the fact that four is an even number and that the dimension of the surface is half this number. These topics are all adequately treated in this part of the book.

IV. Part four (381 pages) is on hypersurfaces. Geodesic lines, lines of curvature, and asymptotic lines are easy generalizations from ordinary surface theory. Curvature however is again quite different. The product of the principal radii is quite different from the product of the principal radii of curvature of a surface in 3-space. The Riemannian curvature of a hypersurface in 4-space

has particularly interesting properties due to the fact that any orientation can be defined in terms of three, which is equal to the number of dimensions of the hypersurface. Many special hypersurfaces are also of considerable interest; among these may be mentioned hyperspheres, ellipsoids, and minimal hypersurfaces. This is only a small indication of the many things treated in this part of the book.

V. Part five, Invariants (120 pages). A book on this subject that makes any claim to completeness must necessarily discuss this subject. The treatment is based on Lie's continuous groups and covers the subject quite adequately.

C. L. E. MOORE

The Logic of Discovery. By R. D. Carmichael. Chicago, The Open Court Publishing Co., 1930. 274 pp.

Thanks to the analyses of both mathematicians and philosophers the science of logic has enjoyed greater development in the last seventy-five years than in the entire twenty odd centuries of its previous history. Up to 1850 it was identified with the formal logic of Aristotle. Since then five generalizations beyond Aristotle have occurred, each one of which has produced a new and more fertile system of logic. These systems are (1) the logic of classes initiated by Boole and Schröder, (2) the logic of relations associated with the names Peirce, Royce, Russell and others, (3) the logic of propositional functions formulated by Russell, (4) the logic of systems or doctrines and system or doctrinal functions, recently developed into an even higher generalization by Sheffer, and (5) the calculus of propositions first formulated by Russell and Whitehead in the *Principia Mathematica*, recently reinterpreted by Wittgenstein, and modified and perhaps even transcended by Hilbert, C. I. Lewis, Weyl, and others.

Carmichael's *Logic of Discovery* is based upon the fourth of these generalizations, the logic of systems or doctrines. Nothing new concerning this logic appears in his book. Its originality consists in its specification of the changes in our conception of mathematics, of reasoning, and of scientific method in the physical and social sciences, which the established concepts of this logic entail.

The well known fact that logical inference in a doctrinal function depends solely on the form is brought into the foreground. The key to this is really the notion of the variable and the discovery of the formal properties of relations which came into the logic of systems from the logic of propositional functions and relations. Carmichael's use of an experiment carried on by Veblen is most convincing on this point. The equivalence of postulate sets for the same system or system function is also indicated. These considerations enable Carmichael to reveal the relativity of the ordinary linguistic statements of a system, and the formal and purely rational side of scientific procedure.

His most original contribution concerns our conception of scientific method. It has been known for a long time by all acquainted with the historical evidence that Bacon's naive theory of induction has very little to do with actual scientific procedure, particularly as it occurs in the most mature sciences. Recently it has become the custom to conceive of all scientific work in terms of the method of hypothesis. But both conceptions have a common presupposition: Scientific statements are considered in terms of a few isolated propositions. The logic of systems and the part which it plays in mathematical physics forces a modifica-

tion of these conceptions. It reveals that bodies of propositions have formal properties and that they can be handled as a whole. Moreover, these formal properties of bodies of propositions prescribe rules which are quite independent of any specific subject matter whatever. In technical language, the properties of the system remain the same if variables are substituted for constants, thereby turning each proposition into a propositional function, and the system or doctrine as a whole into a system or doctrinal function.

Unconsciously science has been taking advantage of this fact. Thus much of scientific procedure consists in establishing identifications between the variables of some purely mathematical system and the physical constants of some empirical subject-matter. Thus instead of working by observation from isolated facts to general principles, or from a hypothesis, put up more or less arbitrarily, to the verification of its deductive consequences, science confronts the actual system of an entire empirical subject-matter with the possible doctrine functions of pure mathematics. Einstein's identification, in the general theory of relativity, of the potentials of the gravitational field with the g_{ik}'s of a tensor form in pure mathematics is a case in point.

This is but one example of the many ways indicated in Carmichael's book in which the modern logic of doctrinal or system functions forces us to modify our conceptions of important things. It is easily the most readable treatment of this aspect of modern logic that has been written.

F. S. C. NORTHROP

Lezioni di Geometria Analitica e Proiettiva; parte seconda. By Annibale Comessatti. Padova, Casa editrice A. Milani, 1931. xii+440pp.+57 figures. Price Lire 60.00.

This second volume of the comprehensive treatise on the basic concepts of analytic geometry is concerned entirely with projective geometry of one, two, and three dimensions, treated analytically and restricted to entities generated by linear forms: points, lines and planes.

While the results obtained in the preceding volume (reviewed in this Bulletin, vol. 36, p. 474) are occasionally cited, it would be possible to read the present one without any knowledge of it further than the general concepts of coordinates and of the elements of projective geometry, treated synthetically.

The general plan of this second volume is much more closely related to our American scheme of instruction than was that of the first. It does not treat a wide range of subjects, but develops the one concept of projective transformations systematically and exhaustively. Following each chapter is a generous list of exercises, and the reader is urged to supplement the algebraic discussion with a free use of synthetic confirmation.

After binary collineations in a purely projective field have been introduced, they are applied to various metrical cases, including pencils of circles and groups of motions. In the ternary field the ordinary problems of projection and section, homology, involutions are first presented; then follow the types of plane collineations, cyclic collineations, and numerous metrical particularizations. Duality and polarity are given an entire chapter, which includes the duality between a bundle and a plane field.

In space of three dimensions more details are given in the development of homogeneous point coordinates; then follow the canonical types of collineations

and the geometric meaning of each. Duality includes the null system, with linear systems of linear line complexes, and the tetrahedral complex.

The last chapter of the book is devoted to a discussion of the foundations of projective geometry from the axiomatic standpoint; it includes postulation, numerical spaces, ordinal relations, harmonic groups, equivalence and the postulate of Archimedes.

This is followed by a development of projective coordinates of imaginary elements, with particular emphasis on the invariance of cross ratios.

The book is excellently printed with black type on thin opaque paper, is practically free from typographical errors, and is provided with a subject index and a list of authors. It provides an adequate and dependable preparation for work in algebraic geometry.

VIRGIL SNYDER

Georg Cantor. By A. Fraenkel. Leipzig, B. G. Teubner, 1930. 78 pp.

In view of present day interest in the foundations of mathematics, this pamphlet is a timely publication. The major portion is naturally devoted to a sketch of Cantor's life and personality, and a full discussion of his development as a mathematician. As might be expected, these are inextricably mingled. Although the *Mengenlehre* holds the center of the stage, some of Cantor's other activities are of no less interest to the reader,—for example, his concern with the Bacon-Shakespeare controversy.

The latter part of the pamphlet is concerned with an outline of Cantor's publications and ends with a bibliography. It is of interest to learn of his contributions to the theory of numbers and the theory of trigonometric series, and instructive to see how these led to the theory of aggregates. It is in precisely such respects that a work of this kind is of value, providing the student with a background that gives a richer knowledge of the subject.

W. A. WILSON

Cours de Mécanique. By Paul Painlevé and Charles Platrier. Paris, Gauthier-Villars, 1929. iv+644 pp.

This book is a printed version of lectures given by C. Platrier (substituting for P. Painlevé) to second-year students at the École Polytechnique. The main topics discussed are Rigid Dynamics, Hydrodynamics, Elasticity, Aerodynamics, Theory of Relativity. The point of view is quite modern and the treatment is very satisfactory. The discussion of the gyroscope is about the best we have seen in a general treatise as is also that on wave-propagation. The discussion of tensor analysis is adequate for a student who is not desirous of being a specialist in this subject. The only criticism that one could fairly make of the work is its very academic character; a book containing several chapters on aerodynamics which does not mention Prandtl's name cannot appeal strongly to the practical man of affairs. Nevertheless the book is a valuable addition to Appell's renowned treatise on Mechanics, which book it (together with Painlevé's *Cours de Mécanique*, Tome I) replaces to a certain extent.

F. D. MURNAGHAN

Theorie und Anwendung der unendlichen Reihen. By Konrad Knopp. Third edition. Berlin, Springer, 1931. 12+582 pp.

The first edition of this book which appeared in 1922 was reviewed in this Bulletin, (vol. 29 (1923) p. 474). Since then there has appeared also an English translation of the second edition.

The earlier chapters of the new edition are in the main identical with those of the first edition, there being improvements in expressions, additional more recent references, and similar changes. Most of the sections which appeared in the first edition which were not of the "notes and remarks" type have properly been elevated to an equality with the remainder of the text by appearing in the larger print. The main changes have occurred towards the end of the book. The last chapter of the first edition dealing with summability of divergent series was rewritten and rearranged in the second edition, giving a well-ordered, general view of the different definitions of summability and a fairly detailed discussion of the Cesàro and Abel processes. The edition under review has been improved by giving a recent very simple proof of the equivalence of the Cesàro and Hölder summability definitions of the same order due to I. Schur and A. F. Andersen, as well as a recent elegant deduction of the Littlewood Theorem that "if Σa_n is Abel summable, that is, $\lim_{x\to 1-0}\Sigma a_n x^n$ exists, and if the na_n form a bounded sequence, then Σa_n is convergent in the usual way", due to Karamata.

The last chapter of the new edition, dealing with asymptotic series, was added originally to the English edition, and is here practically unchanged. Beginning with a careful treatment of the Euler sum formula, the chapter leads up to a satisfying introduction to the notion of asymptotic series and their principal properties. With the addition of this chapter the book has increased its usefulness as a carefully executed and well presented introduction to a very fascinating and important branch of analysis.

T. H. HILDEBRANDT

NOTES

In addition to those of the editorial staff of the Bulletin, the following persons have assisted the editors, either by refereeing papers or by advising concerning papers offered for publication in the present volume: A. A. Albert, Clifford Bell, E. T. Bell, B. A. Bernstein, H. F. Blichfeldt, Daniel Buchanan, A. B. Coble, L. E. Dickson, L. L. Dines, E. L. Dodd, Arnold Dresden, G. C. Evans, Tomlinson Fort, Raymond Garver, L. M. Graves, Lois W. Griffiths, Einar Hille, T. R. Hollcroft, E. V. Huntington, Louis Ingold, A. J. Kempner, E. P. Lane, C. H. Langford, Mayme I. Logsdon, C. C. MacDuffee, W. D. MacMillan, Karl Menger, A. D. Michal, C. N. Moore, F. S. Nowlan, G. Y Rainich, J. F. Ritt, Virgil Snyder, E. B. Stouffer, J. D. Tamarkin, G. T. Whyburn, W. M. Whyburn, K. P. Williams, W. A. Wilson. The editors desire publicly to recognize this service.

The Board of Trustees of the American Mathematical Society has concluded a reciprocity agreement with the Deutsche Mathematiker-Vereinigung similar to those already in force with the London Mathematical Society and the Unione Matematica Italiana. This agreement, which was negotiated through Professor L. Bieberbach, Secretary of the Vereinigung, concerns on the one hand members of the Society resident in the United States and Canada and on the other members of the German organization resident in Germany, Austria, and Danzig. The latter group may become members of the Society by payment of the initiation fee and half the annual dues. They receive the Bulletin for $4.00 annual dues, the Transactions for $6.00, and the two for $9.00. The former group are admitted to the Vereinigung without initiation fee upon the payment of 5 marks annual dues as formerly, but if they take the Jahresbericht, they pay 12 marks, which is half rate. Members of the Society wishing to take advantage of this arrangement may communicate directly with Professor L. Bieberbach, Gelfertstrasse 16, Berlin-Dahlem, Germany.

A new Japanese mathematical periodical, Series I (Mathematics) of the Journal of the Faculty of Science, Hokkaido Imperial University, has been founded recently. The papers in the first two numbers are written in German.

The meeting of the International Physics Union at Brussels on July 12 was attended by representatives of fourteen countries. Professor R. A. Millikan, of the California Institute of Technology, was elected president, as successor to Sir William Bragg. The Union will hold its meeting in 1933 at Chicago, in connection with the World's Fair.

The Bologna Academy of Sciences announces the following subjects for its Adolfo Merlani prize in mathematics, for which competition closes December 31, 1932: (1) a contribution to the study of the roots of the integrals of linear differential equations (in the real domain); (2) a contribution to some question in the mechanics of fluids, preferably in aerodynamics. Competing memoirs should be in Italian, and may be in manuscript or in printed form, provided in the latter case they were not printed before 1930.

A prize has been founded in honor of Eugenio Rignano, under the auspices of the review Scientia, of which he was editor, for a work on *The evolution of the notion of time*. Competing works must be unpublished, or published since 1930; competition closes December 31, 1932.

In commemoration of the one hundredth anniversary of the death of Laplace (in 1827) a memorial monument will be erected in the town of Beaumont, his birthplace. Dedication will take place in the summer of 1932.

Dr. W. S. Adams, director of the Mount Wilson Observatory, has been elected president of the American Astronomical Society.

Dr. G. F. McEwen has been appointed chairman of the newly created committee of the American Geophysical Union on evaporation from free water surfaces.

Professor R. A. Millikan, of the California Institute of Technology, has been made a Knight of the Legion of Honor by the French Government.

Professor R. L. Moore, of the University of Texas, who was appointed by the Council of this Society to the Travelling Lectureship for the current academic year, will give the first series of lectures on the Pacific Coast, on the following dates: University of California at Los Angeles, November 10, 12, 13; Stanford University, November 16, 17, 18; University of Washington, November 20; University of California, Berkeley, November 23, 24, 25. The titles of his lectures are: 1. *Foundations of point set theory*; 2. *Continuous curves*; 3. *Upper semi-continuous collections*. The dates of his lectures in other parts of the country will be announced later. Professor Moore will also give an address entitled *A set of axioms for plane analysis situs*, by invitation of the program committee, at the meeting of the Society to be held at the California Institute of Technology in Pasadena on November 28.

Professor Willem de Sitter, director of the Leyden Observatory, has been elected an honorary member of the American Astronomical Society.

The following have been awarded National Research Fellowships in mathematics for the academic year 1931–32: R. P. Agnew, E. F. Beckenbach, Leonard Carlitz, Benedict Cassen, Max Coral, H. B. Curry, J. L. Dorroh, A. L. Foster, D. H. Lehmer, S. B. Littauer, E. J. McShane, C. W. Mendel, C. B. Morrey, A. L. O'Toole, A. E. Ross, Wladimir Seidel, W. J. Trjitzinsky, Hassler Whitney, S. S. Wilks, Jacob Yerushalmy, Leo Zippin. This list includes renewals.

Arnaud Denjoy, "professeur sans chair" at the University of Paris, has been appointed to the chair of "mathématiques générales."

Dr. Frank Löbell, of the Stuttgart Technical School, has been promoted to a professorship of descriptive geometry.

Dr. Udo Wegner, of the University of Göttingen, has been appointed professor of mathematics at the Darmstadt Technical School.

Dr. R. P. Agnew has been appointed assistant professor of mathematics at Cornell University.

Dr. W. D. Baten has been promoted to an assistant professorship of mathematics at the University of Michigan.

Mr. William Beverly has been promoted to an assistant professorship of mathematics at Lafayette College.

Mr. W. M. Birchby has been promoted to an assistant professorship of mathematics at the California Institute of Technology.

Mr. J. C. Brixey, of the University of Oklahoma, has been promoted to an assistant professorship of mathematics.

Mr. Earl Church, of Syracuse University, has been promoted from an assistant professorship of mathematics to an associate professorship of photogrammetry.

Assistant Professor Tobias Dantzig, of the University of Maryland, has been promoted to an associate professorship of mathematics.

Dr. M. S. Demos, of Columbia University, has been appointed assistant professor of mathematics at the University of New Hampshire.

Professor W. W. Elliott, of Duke University, has been granted Sabbatical leave for the first semester of 1931–1932. He will spend most of this time at Brown University.

Mr. N. C. Fisk, of the University of Michigan, has been appointed dean of the Itasca Junior College, Coleraine, Minnesota.

Dr. H. C. Gossard has been elected president of the New Mexico Normal University.

Dr. P. E. Hemke has been appointed associate professor of mechanics at the Case School of Applied Science.

Dr. Gertrude I. McCain has been appointed professor of mathematics at Marymount College, Salina, Kansas.

Miss Martha P. McGavock has been promoted to an associate professorship of mathematics at Rockford College.

Dr. L. T. Moore has been appointed assistant professor of mathematics at Brooklyn College of the City of New York.

Associate Professor C. S. Porter, of the department of mathematics at Amherst College, has been appointed acting dean of the College.

Mr. H. P. Roges, of the University of Illinois, has been appointed assistant professor of mathematics at Kent State College, Kent, Ohio.

Dr. J. A. Shohat has been promoted to an assistant professorship of mathematics at the University of Pennsylvania.

Dr. L. B. Slichter, of the California Institute of Technology, has been appointed associate professor of geophysics in the department of geology at the Massachusetts Institute of Technology.

The promotion of Professor I. W. Smith to the rank of professor at the North Dakota State College, which was reported without date on page 517 of the July issue of this Bulletin, actually occurred in 1918.

Assistant Professor Pauline Sperry, of the University of California, Berkeley, has been promoted to an associate professorship of mathematics.

Dr. F. G. Williams has been appointed professor of mathematics at Susquehanna University.

Dr. R. C. Yates has been appointed assistant professor of mathematics at the University of Maryland.

The following appointments to instructorships in mathematics are announced:

Duke University, Dr. Ruth W. Stokes;
University of South Dakota, Miss Marjorie L. Heckel;
University of Wisconsin, Dr. Elizabeth Stafford Sokolnikoff.

Mr. Alfred Barnard Basset, F.R.S., known for his work in mathematical physics, died December 5, 1930, at the age of seventy-six.

Professor Ladislaus von Bortkiewicz, of the department of mathematical statistics and insurance at the University of Berlin, died July 16, 1931.

Professor Federigo Guarducci, of the department of geodesy at the University of Bologna, died February 6, 1931.

Professor Ersilia Bisson Minio, of the Reale Instituto Magistrale of Venice, died August 18, 1931.

Richard Bolling Dunn, instructor in mathematics at Yale University, has died recently.

Professor Charles Haseman, of the University of Nevada, died July 9, 1931. Professor Haseman had been a member of the American Mathematical Society since 1907.

Dr. Bessie I. Miller, of the University of Illinois, died February 4, 1931. Dr. Miller had been a member of the American Mathematical Society since 1915.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

334. Dr. A. B. Brown and Professor B. O. Koopman: *Covering of analytical loci by complexes.*

The purpose of this paper is to prove that the locus defined by the vanishing of analytic functions of n variables is a complex of analysis situs. While the truth of this theorem has been recognized more or less generally, no rigorous proof has been given in the past. Van der Waerden has given a proof in the algebraic case, which has no direct generalization. Lefschetz has sketched a proof in the general case by methods which, however, would have to be profoundly altered before the proof could be considered rigorous. The treatment given in the present paper applies both in the real and in the complex domain, both in the small and in the large. In the real case the $(n-1)$-dimensional part of the locus is either vacuous or an orientable $(n-1)$-cycle (mod 2). (Received October 9, 1931.)

335. Dr. Lulu Hofmann: *On the double-point configurations of two projective planes on the same base as the result of the superposition of two distinct projectively related planes of euclidean space.*

The double-point configuration of two projective planes on the same base is thought of as the result of superposing two projectively related distinct planes of euclidean space. The different types of projective relations are classified from the metric point of view, and examined as to their capability of producing the various double-point configurations, when the planes are superposed in all possible ways. The most interesting and important configuration is the so-called "parabolic" one, consisting of one point and one line which are incident. It is shown that this configuration can be produced in $2\,\infty^1$ ways for any non-affine projectivity, in $2\,\infty^2$ ways for a special type of non-similar affinity, and in no way for any other projectivity. For a general non-affine projectivity, the points and lines that can occur as the elements of a parabolic configuration are all the points on two parallels to the vanishing line except their infinite point, and all the lines on a central line conic except the two parallel to the vanishing line. (Received October 9, 1931.)

336. Dr. Wladimir Seidel and Dr. S. B. Littauer (National Research Fellows): *On the lines of Julia of entire functions.* Part I.

In this paper the authors consider the relations between the lines of Julia of an entire function and (1) the asymptotic values attained by $f(z)$ on rays; (2) the order of $f(z)$ on a ray; (3) the type of $f(z)$ on a ray. Among the results obtained we may mention the following two: (I) *If the entire function $f(z)$ attains on the rays $\theta=\theta_1$ and $\theta=\theta_2$ the same finite asymptotic value a, and if the rays $\theta=\theta'$, $\theta_1<\theta'<\theta_2$, and $\theta=\theta^2$, $\theta_2<\theta^2<\theta_1+2\pi$, are lines of Julia, then the two non-algebraic singularities of the inverse function corresponding to the values a attained on $\theta=\theta_1$ and $\theta=\theta_2$ respectively are distinct.* If the type $\mu(\theta_0)$ of $f(z)$ of order ρ on the ray $\theta=\theta_0$ be defined as $\mu(\theta_0)=\limsup_{r\to\infty}(\log|f(re^{i\theta_0})|/r^\rho)$ on the ray $\theta=\theta_0$ we call $f(z)$ of *vanishing type* of order ρ if there exist rays $\theta=\theta_n$ converging toward $\theta=\theta_0$ on which $f(z)$ is of order ρ and positive type and such that $\lim_{n\to\infty}\mu(\theta_n)=0$. We prove (II) *Every ray on which the entire function $f(z)$ is of vanishing type and order ρ is a line of Julia.* (Received October 7, 1931.)

337. Dr. Wladimir Seidel and Dr. S. B. Littauer (National Research Fellows): *On the lines of Julia of entire functions.* Part II.

In this paper the authors consider necessary criteria and sufficient criteria for the existence of lines of Julia of entire functions. They relate the cluster values along rays and a type of continuity to the existence of lines of Julia. Of the results the following one may be mentioned: Let $f(z)$ be an entire function of the variable $z=re^{i\theta}$, and let $\theta=\theta_0$ be a ray on which $f(z)$ does not assume values everywhere dense in the plane. Consider some angle $\theta_0-\delta\leqq\theta\leqq\theta_0+\delta$, $\delta>0$, about $\theta=\theta_0$ and the region R: $1<r<2$, $\theta_0-\delta<\theta<\theta_0+\delta$. Consider the family of functions $f_n(z)=f(\sigma_n z)$ in R, where $\{\sigma_n\}$ is an arbitrary monotonic sequence of positive numbers converging to infinity. Then, the *necessary and sufficient condition that $\theta=\theta_0$ be a line of Julia of $f(z)$ is that at least one of the families of functions $\{f_n(z)\}$, constructed in the manner described above, be non-normal in at least one point of the region R.* (Received October 6, 1931.)

338. Professor J. M. Thomas: *The condition for an orthonomic differential system.*

This paper develops a method of testing whether a given finite set of partial derivatives can be placed in a specified order by assigning integral cotes to independent and dependent variables in Riquier's manner. The treatment involves the discussion of certain systems of linear inequalities. There results a test for determining whether a given system of partial differential equations is orthonomic. (Received September 22, 1931.)

339. Mr. R. M. Foster: *A table of Fourier integrals for practical applications.* Preliminary report.

Closed form evaluations of Fourier integrals have been studied, in the exponential form, preparatory to the compilation of a comprehensive table. This

work has included a search of the literature, direct evaluations to fill in some of the obvious gaps, and a systematic investigation of the domains of the parameters for which the formulas are valid. For the most part, the functions included have been those derivable as special cases of the confluent hypergeometric functions, such as the Bessel functions and the parabolic cylinder functions. The restrictions on the parameters have been stated for the validity of the formulas in terms of Riemann integration; it may prove desirable to extend this to Lebesgue integration. (Received October 10, 1931.)

340. Dr. V. V. Latshaw: *The algebra of self-adjoint boundary value problems.*

The Sturmian boundary conditions associated with a fourth-order differential equation are of two distinct types. There may be three boundary conditions at one point and one at the other, or, as in the case of the vibrating bar, there may be two at each point. H. T. Davis has pointed out that the boundary conditions for the first case can never be self-adjoint, while in the second case only two conditions on the coefficients are necessary for self-adjointness. The purpose of the present study has been to state the corresponding facts for systems of any order. A theorem due to Dunham Jackson was used in calculating explicit conditions for self-adjointness of systems of any order with any admissible number of boundary conditions at each point. Applications to special cases exhibit the power inherent in the formulas derived. (Received October 6, 1931.)

341. Dr. Wladimir Seidel (National Research Fellow): *Note on analytic curves everywhere dense in the plane.*

The author constructs an example of an analytic curve, $w=\sum_{n=0}^{\infty}a_n r^n$, r real, $0\leqq r<1$, a_n complex, having the property that it passes through an everywhere dense set of points in the w-plane. The example is obtained by making use of elliptic functions and conformal mapping. (Received October 6, 1931.)

342. Dr. Lulu Hofmann: *On a certain line locus associated with the lines of two distinct projectively related planes in euclidean space.*

In one of two distinct projectively related planes of euclidean space, a line l is fixed and the planes superposed in all possible ways so that l and its homologue in the other plane coincide. Then the locus Q_l of all lines is determined, which in some one of these superpositions coincide with their homologues. For a non-affine projectivity and a line l not parallel to the vanishing line, nor incident with a focus of its plane, Q_l consists of two parabolas. They are both tangent to l, a certain perpendicular to l, and the vanishing line, and lie on opposite sides of this line. The angle between their axes is bisected by l. In all cases above excluded, Q_l consists completely or partially of pencils of lines. The present problem was suggested by the author in her paper *On a certain metric aspect of plane projective transformations* (this Bulletin, vol. 35, pp. 391–400) as, in a certain sense, the plane dual of the problem there discussed. (Received October 9, 1931.)

343. Professor L. E. Ward: *A third-order irregular boundary value problem and the associated expansion.*

The differential system considered is $d^3u/dx^3+[\rho^3+r(x)]u=0$, $u(0)=u'(0)=u(\pi)=0$, where $r(x)$ is given by a convergent power series in x^3 and is such that, when the radius of convergence of this series is less than π, $r(x)$ is continuous in the interval $0\leqq x\leqq\pi$. An integral equation for $u(x, \rho)$ equivalent to the differential system is used to obtain the asymptotic forms of $u(x, \rho)$ in certain sectors of the ρ-plane for x in the interval $(0, \pi)$. The asymptotic expressions for the characteristic numbers and characteristic functions are then obtained. It is proved that any function equal to an infinite series of the characteristic functions with constant coefficients which converges uniformly in any subinterval of $(0, \pi)$ can be expressed in the form $x^2\phi(x^3)$, where $\phi(x^3)$ is a convergent power series in x^3. In the proof the asymptotic forms of $u(x, \rho)$ in certain sectors of the x-plane for ρ positive and large are found and used, the expansion theorem is proved by use of the Birkhoff contour integral, and an integral equation similar to that for $u(x, \rho)$ is used to obtain asymptotic expressions for certain parts of the integrand. Conditions are given for $x^2\phi(x^3)$ to converge uniformly to $f(x)$. (Received September 18, 1931.)

344. Mr. C. A. Lovell: *Affine geometries which are metric.*

This paper makes use of the theory of the linear dependence of functions to obtain a necessary and sufficient condition for the existence of a tensor of second rank whose covariant derivative, formed with respect to a given connection, is zero, which condition is expressed in terms of the given connection alone. The discussion for the general n is made without restriction as to symmetry. This general result is applied to the case $n=2$ and the symmetric case $n=3$. In addition to this, the paper contains a proof for the positive definite symmetric case $n=3$ of the existence of a system of orthogonal geodesic coordinates. A sufficient condition for a special case is given because it gives rise to the idea of reciprocal surfaces. A condition for $n=2$ that the geometries of a system of paths be projectively equivalent to a Riemannian space is also given. (Received October 8, 1931.)

345. Professor E. D. Meacham: *The determination of the canonical pencils by means of the linear complexes osculating the asymptotic curves of a surface.*

At a point P_x on a surface S, consider the two linear complexes C' and C'', osculating the asymptotic curves on S, and a point P_y in an arbitrary plane (not containing P_x) through the reciprocal polar of the projective normal to S with respect to the quadric of Lie. The null planes of P_y with respect to C' and C'' meet the asymptotic tangents to S in four points which taken in pairs with P_y determine planes the loci of whose null points in C' and C'', as P_y varies, are either planes or quadrics. This paper is principally an investigation of these quadrics. The determination and generation of the canonical pencils are by-products of the results obtained. (Received September 18, 1931.)

346. Professor E. D. Meacham: *The class of surfaces whose osculating ruled surfaces belong to linear complexes.*

By means of Wilczynski's canonical form and local semicovariant tetrahedron of reference, the author in his thesis (Chicago, 1922) studied certain properties, and gave a method of construction, of non-ruled surfaces whose osculating ruled surfaces are contained in linear complexes. In this paper additional properties of this class of surfaces are obtained by using Fubini's canonical form and local covariant tetrahedron of reference. (Received September 18, 1931.)

347. Professor E. D. Meacham: *Certain tetrahedrons associated with a point of a non-ruled surface.*

Wilczynski associated with a point on a surface four osculating linear complexes. This paper is concerned with an investigation of the properties of five skew quadrilaterals, lying on the quadric of Lie, determined by the directrices of the linear congruences common to pairs of these complexes. Four of the quadrilaterals have been previously studied individually, but the properties of the other, and those which the five have in common, are believed to be new. (Received September 18, 1931.)

348. Professor Edward Kasner: *A curiosity in complex geometry.*

The fact that the limit of the ratio of arc to chord for analytic non-real curves is not always unity was discussed by the author in this Bulletin, vol. 20 (1914), p. 524, and the application to relativity theory was given in a note in Nature, December, 1921, p. 434. In the plane a discrete set of values 1.00, 0.94, 0.86, $\cdots$ may be assumed, but in space of three or more dimensions all values are possible. It is now shown that for regular surfaces, in general, the limit in question, for the curves drawn at a regular point, may assume *any* value, real or complex, including 0 and ∞ with the single exception of $3^{1/2}/2 = 0.86$. (Received October 5, 1931.)

349. Professor J. D. Tamarkin: *On the compactness of the space L_p.*

In connection with a recent paper by Kolmogoroff (Göttinger Nachrichten, 1931, pp. 60–63) a set of necessary and sufficient conditions is derived for the compactness of a set of functions defined over n-dimensional euclidean space and integrable together with the pth powers of their absolute values. (Received October 9, 1931.)

350. Professor O. D. Kellogg and Miss Mildred Sullivan: *On the derivatives of potentials on the boundary.*

The current theorems on the existence, as limits, and the continuity, on the boundary, of potentials of simple and double distributions on surfaces, impose heavier conditions on surfaces and densities than are necessary. The present paper is devoted to an exploitation of a condition employed by Dini, which

demands less of these data than the usual Hölder conditions. The results are similar to those given in a paper by Professor Kellogg in the Transactions of this Society, vol. 33 (1931), pp. 486–510, with the difference that there the harmonic functions considered were determined by their boundary values, while in the present case they are determined by the densities of the distributions producing them. (Received October 7, 1931.)

351. Professor Oystein Ore: *Formal theory of linear differential equations.*

The present paper contains a new representation of the formal theory of linear differential polynomials, which is in various ways simpler and more complete than the theories of Blumberg, Loewy, and others. In this paper the main properties of differential polynomials including the decomposition theorem of Landau-Loewy will be considered. It might be observed that the same method applies to all non-commutative domains with one- or double-sided Euclid algorithm. (Received October 7, 1931.)

352. Professor H. M. Gehman: *Concerning sequences of homeomorphisms.*

This paper considers the problem of determining conditions under which a homeomorphism exists between the limiting sets of two sequences of continua, corresponding members of which are homeomorphic. In this connection much use is made of a condition of uniformity of approach. Certain theorems of real-variable function-theory are proved as special cases of the theorems of this paper. The following theorem is typical. Given a sequence (M_i) of closed sets, each member of which contains the following set. Let M denote the set of points common to all the sets of the sequence. If (1) N is a set which is homeomorphic with each set M_i, the homeomorphism being uniform with respect to N, and (2) the set of points corresponding to each point p of N has a sequential limiting point P which is not the sequential limiting point corresponding to any other point of N, then there exists a homeomorphism T such that $T(N) = M$. (Received October 10, 1931.)

353. Dr. T. C. Benton: *Acyclic, bounded, continuous curves which are homogeneous except for a zero-dimensional set having a non-dense perfect* wth *derived set.*

Following the methods of previous papers the author gives constructions for the sets of the type indicated in the title of the paper. There are two main types, depending on whether the branch points of the set are or are not limit points of the set of non-homogeneous points. The constructions are too lengthy to be set out in a short abstract. (Received October 10, 1931.)

354. Mr. Hassler Whitney (National Research Fellow): *A characterization of the closed 2-cell.*

It is shown that if M is a locally connected continuum containing the simple closed curve J such that (1) if ab is any arc in M with just its two end points

on J, then $M-ab$ is not connected, and (2) J is irreducibly homologous to 0 in M (see Vietoris, Mathematische Annalen, vol. 97, p. 464), then M is homeomorphic with a circle and its interior (J going into the circle). In using the word irreducibly, we mean that (2) does not hold in any proper closed subset of M. (2) may be replaced by the condition (2'): Given any $\epsilon>0$ and two points a and b on J dividing J into the two arcs α and β, there exists a set of points a_{ij} in M, $1\leqq i\leqq m$, $1\leqq j\leqq n$, such that we have $a_{1j}=a$, $a_{mj}=b$, $a_{i1}\subset\alpha$, $a_{in}\subset\beta$, Dist $(a_{ij}, a_{i,j+1})<\epsilon$, Dist $(a_{ij}, a_{i+1,j})<\epsilon$, and M is irreducible in this respect. The proof is largely combinatorial, and is in essence fairly simple. (Received October 10, 1931.)

355. Professor J. L. Walsh: *On the overconvergence of certain sequences of rational functions of best approximation.*

Suppose $f(z)$ is an analytic function of z whose singularities form a set E one of whose derivatives $E^{(k)}$ is empty. Suppose C is a closed point set with no point in common with E. Then a sequence of rational functions $r_n(z)$ of respective degrees n of best approximation to $f(z)$ on C such that the poles of $r_n(z)$ lie in E converges to the function $f(z)$ over the entire plane except on the set E. The convergence is uniform on any closed point set containing no point of E, and on any such point set the convergence is better than that of any geometric series. (Received October 28, 1931.)

356. Professor P. A. Smith: *Properties of group manifoids.*

The author obtains some results relative to the structure of n-parameter continuous groups. (Received October 30, 1931.)

357. Dr. Eberhard Hopf: *Proof of the analyticity of the surface of a rotating fluid in equilibrium.*

In this paper we apply some general methods developed elsewhere for the discussion of the analyticity of solutions of non-linear integrodifferential equations to the problem of figures of equilibrium of rotating fluids. The following result, suggested by Poincaré, Liapounoff, and Lichtenstein, is proved: The surface of equilibrium of a rotating homogeneous fluid whose particles gravitate according to Newton's law is analytic at all the points where the apparent force does not vanish. (Received October 31, 1931.)

358. Professor B. A. Bernstein: *On proposition* *4·78 *of Principia Mathematica.*

Whitehead and Russell view proposition *4·78 of their *Principia Mathematica* as a proposition not having its "analogue for classes." The present author (1) proves this view to be a mistaken one by deriving *4·78 from the logic of classes, and (2) shows that the fact of logic for which *4·78 is intended cannot be derived from the *Principia* theory of deduction. (Received October 26, 1931.)

359. Professor Louise D. Cummings: *Hexagonal systems of seven lines in a plane.*

This paper is concerned with the determination of the non-equivalent sets

of seven lines in a plane. Six lines forming a hexagon are selected as the initial six; the addition of the seventh line shows that the number of non-equivalent arrangements is exactly eight. The integers showing the numbers of polygons of $3, 4, 5, \cdots, n$ sides which occur in each of these eight systems are tabulated. The method, developed by Professor H. S. White, of the unique characterization of a line by means of the crossings of contiguous line-segments in a system, has been used to determine the "mark" for every line in the eight systems. The seven marks of a system are employed to prove the equivalence or the non-equivalence of two systems and to determine the substitution connecting two equivalent systems. (Received October 15, 1931.)

360. Professor E. T. Bell: *On the real multiplication of elliptic functions.*

An explicit solution of the difference equation which occurs in the real multiplication of elliptic functions, and which is equivalent to the problem of real multiplication, is given. This is the equation to which Cayley devoted several papers without arriving at a solution. The equation is in two variables with variable coefficients. It is found that the variables are separable, and thereby the complete solution is easily attainable. This paper will appear in the Annals of Mathematics. (Received October 16, 1931.)

361. Professor W. M. Whyburn: *On the integration of unbounded functions.*

In a former paper in this Bulletin (vol. 37, pp. 561–564) the author established a relation between the integrals of F. Riesz and Lebesgue. The present paper extends the main theorem of the former paper to obtain a necessary and sufficient condition for the measurability of an unbounded function. A new necessary and sufficient condition for Lebesgue summability is given. A definition of an integral of an unbounded function is given that, although similar to the definitions of Harnack and Denjoy, seems to apply to functions not covered by former definitions. (Received October 23, 1931.)

362. Dr. D. H. Lehmer: *On the distribution of roots of polynomials with integer coefficients.*

This note is concerned with polynomials of the form (1) $x^n+a_1x^{n-1}+\cdots$, where $n>1$, and where the a's are integers. It is assumed that either (1) has no complex roots, or all its roots lie on the same circle in the complex plane. A new proof is given of Kronecker's elegant theorem that in case this circle is the unit circle, then the roots of (1) are roots of unity. Several consequences of this theorem are given, for example: Every irreducible equation of the type (1) with no complex root must have a root outside the open interval $(-\sqrt{2}, \sqrt{2})$. In case all the roots of (1) lie on a circle, its radius is of the form $k^{1/\delta\epsilon}$ where k is an integer, δ is the G.C.D. of the subscripts of the non-zero coefficients of (1) and ϵ is 1 or 2 according as n is odd or even. A necessary and sufficient condition for the arguments of the roots to be commensurable with 2π is given. (Received October 29, 1931.)

363. Dr. D. H. Lehmer: *On certain cyclotomic functions.*

In this paper we study the properties and interrelations of three cyclotomic functions introduced by Schur, Pierce and Poulet. Each is a function of an integer n and a polynomial f with integer coefficients. It is shown that the classes of numbers represented by each of the first two functions are the same. By a combination of the methods employed by Schur and Pierce it is possible to improve the results of each. In case the polynomial f is symmetric, the values of Pierce's function are perfect squares. The roots of these squares turn out to be the values of a function which includes Poulet's function as a special case. This function bears the same relation to Pierce's function that our extension of Lucas' U_n does to x^n-1. Each function is found to form a recurring series so that one may easily calculate the values of the functions for large indices n. Conditions on f are found for which corresponding functions increase very slowly. This gives a practical method for the identification of large primes. (Received October 29, 1931.)

364. Mr. J. L. Botsford: *Contributions to the theory of multiple-point integral invariants.*

This paper deals with necessary and sufficient conditions for multiple-point integral invariants (A. D. Michal, Transactions of this Society, July, 1927) of the system $dx^i/dt=\xi^i(x^1, \cdots, x^n)$. Previous investigations of projective integral invariants (A. D. Michal, this Bulletin, June, 1931) are extended to multiple-point integral invariants. (Received October 30, 1931.)

365. Mr. A. H. Clifford: *Continuity and differentials of implicit functionals.*

Sufficient conditions have been given (Volterra, *Fonctions de Lignes*; proved in detail by G. C. Evans, Colloquiun Series, Vol. V., Part I.) that a functional equation of the type $H^x[\phi^t, f^t]=0$ (where H is a functional of the two continuous functions ϕ^x, f^x and depends continuously on the real variable x) defines ϕ^x implicitly as a functional of f. In this paper the continuity of the implicit functional thus defined is established, and, with additional hypotheses regarding the differentials of H, the implicit functional is shown to admit Frechet differentials of a special type (Schmidt integral power forms) out to any order r, or of all orders. These results are immediately applicable to the inverse of a functional transformation, i.e., to the solution $\phi^x[f^t]$ of an equation of the type $f^x=F^x[\phi^t]$. (Received October 30, 1931.)

366. Professor Raymond Garver: *Concerning certain root location theorems.*

This paper is concerned with relations existing between the roots of an algebraic equation and those of its derived equation. First, a number of known results, including the Gauss-Lucas theorem, Jensen's theorem, and certain results of Walsh, Nagy and Mitchell are proved with the aid of simple transformations. The possibility of employing transformations in theorems of this type has been mentioned a few times in the literature, but the method has not been used extensively. Second, a new type of problem is contemplated, namely the

determination of the maximum value of the ratio of the area of the smallest convex polygon enclosing the roots of the derived equation to the area of the smallest convex polygon enclosing the roots of the given equation. For quartic equations with real coefficients this ratio is always less than $3/4\sqrt{2}$, but can be made arbitrarily close to this value. (Received October 30, 1931.)

367. Mr. R. S. Martin: *The explicit determination of the most general linear continuous functional transformation preservative of continuity.*

The Stieltjes kernel of the most general linear continuous functional transformation which transforms continuous functions of a single variable into continuous functions of a single variable is completely characterized. Extensions to the transformations of functions of many variables are also considered. Explicit expression for the transformation of the kernel of a continuous linear form under functional transformations preservative of continuity is given together with sufficient conditions that expression be possible in terms of Riemann-Stieltjes integrals. (Received October 30, 1931.)

368. Mr. R. S. Martin: *A theorem of the limits of Stieltjes integrals.*

The writer proves by elementary methods that if a sequence of functions $y_n(x)(a \leq x \leq b)$ each integrable Riemann-Stieltjes with respect to a function $\alpha(x)$ of limited variation on (a,b), converges in any manner to a function $y(x)$ also integrable with respect to $\alpha(x)$, then $\lim_{n=\infty} \int_a^b y_n(x)d\alpha(x) = \int_a^b y(x)d\alpha(x)$. (Received October 30, 1931.)

369. Mr. R. S. Martin: *Representation of function valued polynomial functionals of a continuous function.*

The writer considers the representation by multiple Stieltjes integrals of function valued polynomial functionals of a continuous function of a single variable. The Stieltjes kernel corresponding to the most general polynomial which for continuous argument is a continuous function is completely characterized. (Received October 30, 1931.)

370. Mr. R. S. Martin: *Function valued analytic functionals of several functions.*

Function valued analytic functionals of several continuous functions are defined by means of their expansion in a series of multiple Stieltjes integrals. A notation whose manipulation is analogous to that of umbral indices is introduced and the processes justified. Application of the results is made in the solution of completely integrable systems of total functional-differential equations. (Received October 30, 1931.)

371. Mr. R. S. Martin: *Contribution to the theory of analytic functions in abstract spaces.* Preliminary report.

The writer employs a modification of Banach's postulational definition of an abstract vector space, replacing the condition of closure under multiplica-

tion by real numbers by the condition of closure under multiplication by numbers of a given ring. Some properties of analytic correspondences between the elements of such spaces are studied relative to the choice of the ring. In particular, if the space is closed under multiplication by all complex numbers, analytic correspondences have properties analogous to those of ordinary analytic functions which they include as a special case. (Received October 30, 1931.)

372. Mr. Harry Matison and Mr. A. H. Levine: *Integral addition theorems for Bessel functions.*

The Cauchy problem of the equation $\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 - u = 0$ gives rise to a pair of functional transformations on two functions. These transformations involve Bessel function of orders zero and one. The fact that these transformations form a continuous group when t is regarded as a parameter leads to identities involving repeated intergrals of Bessel functions and two arbitrary functions. After a series of transformations, these are found to imply a set of twelve identities involving only Bessel functions and integrals of Bessel functions. Finally this set of twelve identities is shown to be equivalent to the fact that the transformations considered form a group. (Received October 30, 1931.)

373. Professor A. D. Michal: *Parallel displacement of a multiple-point tensor along several curves in Riemannian geometry.*

The notion of a multiple-point tensor was introduced by the author in connection with his investigations on functional invariants (Transactions of this Society, 1927, pp. 612–646). The present paper deals with a certain parallel displacement of a multiple-point tensor along several curves in Riemannian geometry. This definition reduces to that of Levi-Civita for the case of the one-point tensor (usual type of tensor). (Received October 31, 1931.)

374. Professor O. W. Albert: *Relations between the projective and metric differential geometries of surfaces.*

In this paper we compare the Gauss equations of metric geometry for a special oblique system of axes with the differential equations of projective geometry of a surface used by Professor Wilczynski (Transactions of this Society, vol. 8) and determine relations between the homogeneous coordinates of a point and its Cartesian coordinates referred to our special trihedral. We get simpler metric equations of the osculating linear complexes of the asymptotic curves, C' and C'', given first by Morrison (American Journal of Mathematics, vol. 39). We then find similar equations for the osculating linear complexes of the osculating ruled surfaces, C_1 and C_2, and determine the directrices of their congruence and the conditions for these directrices to be real, to be parallel to or perpendicular to the surface normal. We show that the directrices of the congruence of C' and C'', the directrices of the special congruence of C_1 and C_2, and the tangents to the asymptotic curves are the six edges of a tetrahedron.

By studying the other congruences determined by the four complexes, we find the metric conditions on the invariants of C_1 and C_2 which make the directrices real, imaginary or coincident. We also find the metric conditions for the directrices of the congruences of the four complexes to determine a linear

congruence whose directrices are the two lines common to the four complexes. (Received October 31, 1931.)

375. Professor A. A. Shaw: *An overlooked numeral system of antiquity.*

In this paper the author gives an account of the Armenian alphabetic numerals used in the fourth and subsequent centuries of our era. The Armenians used the first nine letters of their alphabet to represent units, the second nine to represent tens, the third nine to represent hundreds, and the fourth nine to represent thousands. This arrangement agrees with the Semitic and the Greek alphabetic systems of numeration. But the works of Ananiah Shiragooni (first quarter of the seventh century, A.D.) reveal *variant forms* of numbers involving multiplicative and additive principles which reduce the thirty-six symbols to twelve and place the Armenian notation on a *decimal* basis without use of zero. The corresponding variant forms do *not* exist in the Semitic and the Greek systems. The paper closes with a few illustrations involving the four fundamental operations. (Received October 30, 1931.)

376. Professor A. R. Williams: *A transformation of the lines of space effected by means of two quadratic reguli.*

If we take two quadratic reguli which do not lie on the same quadric, and two generators of each regulus of general position, the Plücker coordinates, p_{ik}, of either of the two transversals of these four generators are cubic functions of the coordinates of the other. The special lines which do not have unique transforms constitute a congruence. The use of a quartic scroll in the same manner suggests itself, but leads only to a linear transformation on the line coordinates, since the generators of the scroll belong to a certain linear complex. (Received November 2, 1931.)

377. Professor B. C. Wong: *Enumerative properties of r-space curves.*

In the study of enumerative properties of algebraic curves in space of r dimensions it is often convenient to decompose a curve into two or more component curves and to study these curves both individually and in relation to one another. In this paper the writer makes a systematic study of the skew m-sided polygon with $m-1+p$ vertices, or the connected m-line with $m-1+p$ incidences into which a curve C_p^m of order m and deficiency p has been completely decomposed, and obtains numerous new results for C_p^m, some of which are difficult to obtain otherwise. (Received November 2, 1931.)

378. Professor Lincoln La Paz: *Variation problems of which the extremals are minimal surfaces.*

In this paper it is shown that the partial differential equation $F(x,y,z,p,q,r,s,t)\equiv(1+q^2)r-2pqs+(1+p^2)t=0$ admits a multiplier $M(x,y,z,p,q)\neq 0$, unique up to a constant factor, for which the equivalent equation $MF=0$ has a self-adjoint equation of variation. This result, in sharp contrast to that obtained by Darboux in his study of the analogous problem for the ordinary differential

equation $y''-\phi(x, y, y')=0$, is shown to be due to the special form of the function F. The most general problem of minimizing $I=\iint f(x, y, z, p, q)dx\,dy$ associated with the equation $MF=0$ is found to have the integrand function $h=(1+p^2+q^2)^{1/2}+\partial w_1/\partial x+\partial w_2/\partial y$ where w_1 and w_2 are arbitrary differentiable functions of x, y and z alone. (Received November 4, 1931.)

379. Dr. Ben Dushnik: *On the Stieltjes integral.*

This paper presents the results of studying the application of a general limiting process (as first defined explicitly by E. H. Moore and H. L. Smith, *A general theory of limits*, American Journal of Mathematics, vol. 44, 1922, pp. 102–121) to an additive many-valued number-valued set function. An example of such a set function would be the sums considered in the classical definition of the Riemann and Stieltjes integrals; additional examples are found in this paper. The limit of such a set function when it exists (as a finite number) is called the *integral* of the set function. The theorems developed include one on integration *by parts*, thus solving a problem of Pollard (*The Stieltjes integral and its generalizations*, Quarterly Journal of Mathematics, vol. 49, 1920–23), and several which concern themselves with sequences of set functions. When the additional examples of set functions are considered, there are demonstrated a few existence theorems, as well as certain other theorems whose main trend is to generalize some of the known properties of the Stieltjes integral. (Received November 3, 1931.)

380. Professor C. N. Moore: *On the absolute convergence of the developments in Bessel's functions.*

The convergence and uniform convergence, summability and uniform summability of the developments in Bessel's functions have been studied by various writers. The case of absolute convergence, however, seems to have received little or no attention. In the present paper certain asymptotic formulas for the coefficients of these developments, obtained by the writer in a series of previous papers, are used to derive various sets of sufficient conditions for absolute convergence. As would naturally be expected, it is found that somewhat more stringent requirements are needed than in the case of Fourier series. (Received November 5, 1931.)

381. Professor W. D. Baten: *Frequency laws for the sum of n variables which are subject each to given frequency laws.*

This article develops frequency laws for the sum of n independent variables which have known frequency laws. These functions for the sum are derived by Dodd's method. Relations between measures of precision for the sum function and the measures of precision for the individual frequency functions are treated. A certain definite integral, which arises in developing frequency laws for the sum, is evaluated by integrating around a certain contour in the complex plane. Certain frequency functions are expanded into finite series similar to Charlier's series. A type 3 curve is obtained for the law for the sum of the squares. (Received November 3, 1931.)

382. Professor H. T. Davis: *Polynomial approximation by the method of least squares.*

In this paper the author considers the problem of fitting polynomials from the first to the seventh degree inclusive to data given over a discrete range from $x=-p$ to $x=+p$. The method of finding polynomials which are orthogonal for discrete summation instead of continuous integration is employed and an explicit set of polynomials is calculated. These are shown to agree essentially with polynomials previously found by J. P. Gram (Journal für Mathematik, vol. 94, 1883, pp. 41–73). Some properties of these polynomials are exhibited and elaborate tables are computed which facilitate the practical application of the methods. This paper is an extension of one on a similar subject published by Dr. V. V. Latshaw and the author in the Annals of Mathematics, vol. 31 (1930), pp. 52–78. (Received November 4, 1931.)

383. Professor Saul Pollock: *On the determination of formulae for the orders of line loci in five-space by graphic representation.*

Graphic representation (introduced by B. C. Wong, *On the number of apparent triple points of surfaces in space of four dimensions*, this Bulletin, vol. 35, 1929, pp. 339–343), is a method by decomposition based on the principle of preservation of number. By means of specialized Cartesian systems for four-space and graphic systems for five-space, the Wong method may be extended to the determination of the orders of the ruled loci in five-space whose elements are incident with given curves, surfaces, and hypersurfaces, and satisfy certain conditions. The purpose of this paper is to show how these formulae may be derived by an extension of the Wong theory. (Received November 6, 1931.)

384. Professor Tibor Radó: *On convex regions.*

If the interior of a circle is mapped in a one-to-one and conformal way upon a convex region, then every circle interior to this circle is also carried into a convex curve, according to a theorem of Study. This suggests an investigation of topological transformations carrying circles into convex curves. Every transformation which is sufficiently regular from the point of view of differentiability has this property in the small, as may be easily shown. If a transformation has this property in the large, then it has on the boundary continuity properties similar to those of conformal transformations. (Received November 4, 1931.)

385. Professor R. L. Wilder: *On the linking of Jordan continua in E_n by $(n-2)$-cycles.*

The following is a classical theorem in the theory of Jordan continua (Cf. R. L. Moore, this Bulletin, vol. 29, pp. 295–296; C. Kuratowski, Fundamenta Mathematicae, vol. 6, pp. 140–141): If, in the plane, two points P and Q are separated by a Jordan continuum M, then P and Q are separated by a simple closed curve of M. In the present note it is shown that if in E_n a Jordan continuum M is linked by a cycle Γ^{n-2}, then Γ^{n-2} links a simple closed curve of M. For the case $n=2$, the former theorem follows as a special case. (Received November 4, 1931.)

386. Professor G. Y. Rainich: *Covariant form of a special case of Dirac's equations.*

It is known that Dirac's equations of the theory of the electron, although they do not appear in covariant form, are invariant under the group of Lorentz transformations. The independent variables in these equations—the ψ's—are not components of a tensor, but certain quadratic expressions in the ψ's are. The general problem of which a special case is treated here is to write a system of equations in second degree quantities equivalent to the Dirac equations and to put them in covariant form. The problem is complicated by the fact that the ψ's are not uniquely determined by the second degree quantities. The special case considered is characterized by stating that the ψ's are equal in pairs, or, that the two invariants and the six-vector vanish, and the two four-vectors coincide. The second degree quantities reduce to the components of one vector j of zero square. Complicated differential relations for its components are found by painful computation; and then, using the invariance of the system, comparatively simple differential equations are derived in covariant form. They are *non-linear* in the components of the vector j, so that Dirac's equations may be considered as results of linearization of our system. (Received November 4, 1931.)

387. Professor Cornelius Lanczos: *The electromagnetic field as a natural property of Riemannian geometry.*

Einstein's field-equations $R_{ik}=0$ are not to be considered as the simplest characterization of a Riemannian geometry if we require that this geometry should have no singularities. This restriction leads to euclidean geometry only. To generalize these equations we must expect to find field-equations for R_{ik}. Supposing that these equations obey a variation principle, we can prove generally that they can be integrated by a free vectorial function ϕ_{ik}. This vector ϕ_{ik} is determined by the divergence-condition for R_{ik} which must be fulfilled in addition to the field-equations. Selecting the Lagrangian function by the condition that the varied field-integral should not depend on the arbitrariness of the unit of length, we can show that the vectorial function ϕ_{ik} has the same properties as the vector potential of the electromagnetic field, and we can deduce also the motion-law of Lorentz for an electric particle.. (Received November 3, 1931.)

388. Miss Grace Shover: *Class number in a linear associative algebra.*

Consider a semi-simple algebra with a principal unit taken over the rational field. If β is any number in the left ideal $\mathfrak{R}$, there exists an integral matrix M such that $S(\beta)=MG$ where $S(\beta)$ is the second matrix of β and G is a matrix representation of $\mathfrak{R}$. The adjoint M^A of M belongs to the same minor class as G. Using a theorem of Minkowski, we can find a number β of $\mathfrak{R}$ such that the determinant of the corresponding M^A is bounded by a number which depends only on the algebra. Hence the class number is finite. If G is a left (right) ideal matrix with class matrices D_p, then $G^{AT}T^A(G^{AT}T^{AT})$ is a right (left) ideal matrix with class matrices D_p^T where G^{AT} is the transpose of the adjoint of G and T is the discriminant matrix of the algebra. Hence the right and left class numbers are equal. (Received October 5, 1931.)

389. Dr. L. E. Bush: *Note on the discriminant matrix of an algebra.*

C. C. MacDuffee (Transactions of this Society, vol. 33, p. 427) defines a *normal basis* for a linear associative algebra with a principal unit. In this paper an extension of this definition to a general linear associative algebra is made, and its existence for such an algebra is proved. MacDuffee's theorem 2, which gives a simple relation between the multiplication constants of an algebra with a normal basis, is shown to hold for a general linear associative algebra. (Received October 20, 1931.)

390. Dr. L. E. Bush: *On the equivalence of the decompositions of an algebra with respect to a principal idempotent.*

A definition is given for the *equivalence* of the decompositions of a linear associative algebra with respect to two different principal idempotents, and from this definition a necessary and sufficient condition for two such decompositions to be equivalent is deduced. Formulae are found which give all the principal idempotents of certain classes of algebras, and it is shown that for an algebra belonging to one of these classes the decompositions with respect to any two principal idempotents are equivalent. A *semi-modulus* of an algebra is defined and it is shown that the above mentioned property is possessed by every algebra with a semi-modulus. (Received October 20, 1931.)

391. Professor W. E. Roth: *On algebraic equations having only real roots.*

The present paper proves the following theorem for equations $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n=0$, $a_0a_n\neq 0$, whose roots x_i, $i=1, 2, \cdots, n$, are all real. If $s_k=x_1^k+x_2^k+\cdots+x_n^k$, where k is any integer, and if $a_{ij}^{\lambda}=s_{\lambda(i+j-2)}$, $2\leq i+j\leq 2h\leq 2n$, where h is a positive integer and λ any integer not zero, then the determinants $\left|a_{ij}^{\lambda}-x^{\lambda(i+j-2)}\right|$ are negative only on intervals which do not contain roots of $f(x)=0$. Some extensions of this theorem to the location of simple and multiple roots of $f(x)=0$ are developed. (Received October 16, 1931.)

392. Professor R. V. Churchill: *Canonical forms for symmetric linear vector functions in pseudo-euclidean space.*

Any symmetric matrix has a simple canonical form under orthogonal transformations. Geometrically this form is a canonical form for the symmetric linear vector function or the symmetric second rank tensor in euclidean space. It seems that canonical forms have not been found for matrices under pseudo-orthogonal or Lorentz transformations. This is done in this paper for certain matrices with four rows. Geometrical terminology and methods are used. Symmetric linear vector functions in the space of special relativity are classified here into four types according to the values of their invariants, and a canonical form is found for each type. The method used to find these forms is essentially that of finding sets of orthogonal directions whose vectors are transformed in a simple way by the function, and such that when referred to these directions the function can be described by using a minimum number of its independent

components. Certain properties of the function are made evident by these forms. (Received November 4, 1931.)

393. Mrs. Pearl C. Miller: *Four points of intersection of lines drawn from the vertices of a triangle to the points of contact of the nine point circle with the in-and ex-circles.*

This paper by means of trilinear coordinates finds the equations of the lines mentioned in the title and the coordinates of their points of intersection, also the equations of the pencil of lines from a single vertex to the four points of the harmonic range, and thus the following theorems are established. Theorem 1: If lines are drawn from the vertices of a triangle to the points of contact of the opposite ex-circles with the nine point circle, and also a line is drawn from the center of the nine point circle to its point of contact with the in-circle, these four lines meet in a point which is the fourth harmonic to the nine point center, the in-center and the point of tangency of the nine point circle with the in-circle. Theorem 2: A point having similar properties lies on each of the lines drawn from the nine point center to the points of tangency of the three ex-circles with the nine point circle. (Received October 26, 1931.)

394. Professor Arnold Emch: *The symmetric (n,n)-correspondence and some geometric applications.*

This paper shows how by means of the completely symmetric or involutorial (n, n)-correspondence a large number of geometric problems involving the property of closure may be solved in a manner which is very simple when compared with the arduous labors which are often necessary to accomplish the same result by synthetic means. Among the curves and surfaces studied by this method are the Lüroth quartic, admitting of an infinite number of inscribed pentalaterals, and the Schur sextic in space which admits of ∞^1 inscribed pentahedrons whose faces envelope a class-cubic and whose edges generate a ruled octavic, its connection with the Lüroth quartic being considered. It is also proved that the Schur sextic is the Jacobian curve of a net of quadrics and that its manifold is ∞^{20} as compared with ∞^{24} for the general sextic of genus 3 (or 4). (Received October 28, 1931.)

395. Professor J. M. Thomas: *Riquier's existence theorems.*

This paper is an addendum to one which appeared previously under the same title (Annals of Mathematics, (2), vol. 30, 1929, pp. 285–310). It indicates how to form the integrability conditions of an orthonomic system one or more of whose left members are derivatives of other left members, a detail omitted from the previous exposition. In addition, the process of forming the complete set of monomials is modified so that the complete set always contains the original set, a property possessed by the complete set as defined by Janet. The process indicated in the present paper, however, still has the advantage of forming the complete and complementary sets directly from the Maclaurin expansion. Finally, Riquier's method of reducing an orthonomic system to passive form is applied to deduce the general theorem on compatibility of systems of total differential equations given by O. Veblen and the author (Annals of Mathematics, (2), vol. 27, 1926, p. 290).

396. Professor M. A. Basoco: *Fourier developments for certain pseudo-periodic functions in two variables.*

The functions $\phi_{\alpha\beta\gamma}(x, y) \equiv \theta_1' (\theta_\alpha(x+y)/\theta_\beta(x)\theta_\gamma(y))$, where the θ's are Jacobi theta functions and (α, β, γ) are certain sixteen triads out of the possible ones which can be selected from the numbers 0, 1, 2, 3, have been studied by Jacobi, Kronecker, Hermite and others. More recently, Bell (Transactions of this Society, 1921; Colloquium series, VII, p. 88) has emphasized their importance in connection with certain arithmetical results. In the present paper we are concerned with certain functions $\Theta_{\alpha\beta\gamma}(x, y)$ related to the above functions by the identity $\Theta_{\alpha\beta\gamma}(x, y) = \theta_1' / \theta_\beta(x) \cdot \phi_{\alpha\beta\gamma}(x, y)$. The arithmetized Fourier developments of these functions are obtained in a manner suitable for use in certain problems in number theory. These developments lead to the consideration of incomplete numerical functions in two variables. So far as the author is aware no expansions involving this type of functions have hitherto appeared in the literature. However, Uspensky (this Bulletin, October, 1930) by purely arithmetic methods has proved certain formulae of the paraphrase type which have reference to incomplete functions in more than one variable. (Received October 21, 1931.)

397. Dr. I. J. Schoenberg: *On finite and infinite completely monotonic sequences.*

F. Hausdorff (Mathematische Zeitschrift, vol. 16, pp. 220–248) has proved the following theorem: The infinite sequence of numbers $\mu_0, \mu_1, \mu_2, \cdots$, is completely monotonic, i.e., the whole set of inequalities

$$(1) \qquad \Delta^k \mu_l \equiv \mu_l - \binom{k}{1}\mu_{l+1} + \cdots + (-1)^k \mu_{l+k} \geqq 0, \quad (k, l = 0, 1, 2 \cdots),$$

holds, if and only if there is a non-decreasing function $\chi(t)$ for $0 \leqq t \leqq 1$, such that (2) $\mu_n = \int_0^1 t^n d\chi(t)$, $(n = 0, 1, 2, \cdots)$, holds. The author gives a new proof for this theorem. In order to find the parametric representation (2) for infinite sequences, the analogous problem for a finite sequence $\mu_0, \mu_1, \cdots, \mu_p$, is first solved. Such a sequence is also called completely monotonic if $\Delta^k \mu_l \geqq 0$ for $k+l \leqq p$. A parametric representation is easily found for the most general solution of this finite set of linear inequalities which, on making p become infinite, yields the parametric representation (2) for the original transcendental problem. The same method permits us to assign a rather large class of infinite sets of linear inequalities of the same type as (1), whose solutions may be represented parametrically by means of Stieltjes integrals. This generalization will be considered in another paper. (Received November 5, 1931.)

398. Professor E. W. Chittenden and Mr. Selby Robinson: *Properties of coverings of abstract type related to reducibility.*

Chittenden (Transactions of this Society, vol. 31, pp. 290–321) has developed a theory of the reducibility of coverings of abstract type. The present paper extends and perfects this theory with respect to the reducibility of a single covering. The notation and terminology are improved, and new results are found for the case of coverings of irregular power and of regular powers which

are limits of lower powers. New sets of equivalent properties of coverings are found. These results are applied by Robinson in the study of classes of coverings. (Received November 5, 1931.)

399. Mr. R. M. Foster: *The linear graphs associated with electrical circuits.*

If all the electrical properties are abstracted from a given electrical circuit, there remains a geometrical circuit, or linear graph, completely characterized by the sets of branches terminating at the various vertices. To meet the conditions of electrical problems the concept of equivalent graphs is extended to include cases where two graphs become homeomorphic upon the interchange of any branches or groups of branches which have a series connection. Enumerations of distinct graphs are given, classified according to nullity (number of branches minus number of vertices plus number of separate parts) and according to rank (number of vertices minus number of separate parts). A study is made of symmetrical graphs, defined as non-separable, regular graphs in which all of the branches, as well as all of the vertices, enter symmetrically. The graphs formed by the edges of the regular polyhedra are symmetrical; many other cases are shown to exist. (Received November 4, 1931.)

400. Professor V. W. Adkisson: *On extending a continuous (1–1) correspondence of continuous curves on a sphere.*

A necessary and sufficient condition that a continuous $(1-1)$ correspondence of two plane continuous curves be extendible to a correspondence of their planes has been given by H. M. Gehman, (Transactions of this Society, April, 1926, pp. 252–265). By a slight modification of the definition of "sides preserved" Gehman's result is extended to continuous curves on a sphere. The following theorem is also established. If M is a cyclicly connected continuous curve lying on a sphere S, and T a continuous $(1-1)$ correspondence such that $T(M)=M$, then a necessary and sufficient condition that T be extendible to S is that sense be preserved (or altered) on the simple closed curves of M. Sense is defined for simple closed curves on the sphere to conform with the definition for sense in the plane (J. R. Kline, Annals of Mathematics, vol. 19, pp. 185–200). (Received November 2, 1931.)

401. Professor L. W. Cohen: *On the minors of absolutely convergent determinants.*

Let $\Delta=|\delta_{ik}+a_{ik}|$ be an infinite determinant such that

$$\sum_{i=1}^{\infty}|a_{ii}|<+\infty;\ [\sum_{i=1}^{\infty}[\sum_{k=1}^{\infty}|a_{ik}|^q]^{p/q}]^{1/p}=\sigma<+\infty\ (i\neq k);$$
$$1/p+1/q=1;\ 1<p\leqq 2.$$

Let $A_{r_1\cdots r_s;\ c_1\cdots c_s}$ be the minor of Δ obtained by striking out the rows r_i and the columns $c_i (r_i\neq c_i;\ i=1, 2, \cdots, s)$. Let $[\sum_{k=1}^{\infty}[\sum_{i=1}^{\infty}|a_{ik}|^p]^{q/p}]^{1/q}=\tau$. Certain bounds for the minors are obtained, typical of which is

$$[\sum_{c_1\cdots c_s=1}^{\infty}[\sum_{r_1\cdots r_s=1}|A_{r_1\cdots r_s;\ c_1\cdots c_s}|^q]^{p/q}]^{1/p}\leqq s!\sigma^s P(1+\tau/(1-\sigma))^s$$

where $r_i\neq c_i$, $P=\pi(1+|p|)$, p being a circular product $a_{i_1i_2}a_{i_2i_3}\cdots a_{i_mi_1}$, and the product is extended over all such p's. (Received November 4, 1931.)

402. Professor H. H. Conwell: *Linear associative algebras of infinite order whose elements satisfy finite algebraic equations.*

In this paper an investigation is made of linear associative algebras of infinite order, whose elements satisfy finite algebraic equations with coefficients in the field Ξ over which the algebra is taken. Assuming the existence in the algebra of a set of elements U of such a nature that for every $a \neq 0$ there is determined uniquely a positive integer n and a set of distinct elements $u_1, u_2, u_3, \cdots, u_n$ of U and n non-zero elements $\xi_1, \xi_2, \cdots, \xi_n$ of Ξ such that $a = \sum_{i=1}^{n} \xi_i u_i$, it is shown that many of the theorems of finite linear associative algebras can be extended to non-finite algebras. (Received October 14, 1931).

403. Mr. Michael Goldberg: *The isoperimetric problem for polyhedra.*

The problem of the determination of the convex polyhedra of greatest volume for a given number n of faces and a given surface area has been treated by Steiner, Lindelöf, Minkowski, Steinitz (Crelle's Journal, vol. 158, 1927, and vol. 159, 1928), but solutions have been obtained only for $n < 8$. Lindelöf proved that a necessary condition is that the faces of the polyhedra be tangent to a sphere at the center of gravity of the faces. Let $K = F^3/V^2$ where F = surface, V = volume. This paper establishes upper and lower bounds for K, limiting K to a narrow band of values. Series of polyhedra for which the values of K lie between these bounds are given. An example of a lower bound for K is $K \geq 6(n-2) \tan A(4 \sin^2 A - 1)$, where $A = \pi n/(6(n-2))$. The equality holds for the regular tetrahedron, cube and dodecahedron, showing that they are "best." The regular octahedron and icosahedron are not the "best," and indeed, the paper exhibits "better" polyhedra. (Received September 8, 1931.)

404. Professors Virgil Snyder and Hazel Schoonmaker: *Two involutorial transformations, of orders* 11 *and* 9, *associated with null reciprocities.*

The first transformation discussed by the authors is defined by means of 4 skew lines a_i, a fixed point A_0 not on any a_i, and 5 fixed lines of a pencil g. A plane ω through A meets a_i in A_i and contains a unique point Q such that the 5 lines QA_i are projective with g. As ω describes the bundle A_0, Q describes a cubic surface $F_3 : A_0 A a_i$. If A_0 describes a line u, F_3 describes a pencil. Any point y of space determines a point A_0 on u, and the line $A_0 y$ meets the associated F_3 in y'. The involution (y, y') is of order 11. Its complete table of characteristics is determined. In the second transformation any plane through u determines a conic having a given cross ratio at $A_1 \cdots A_4$. The transversal through y meeting u and a sixth line u' meets the conic in y'. This involution of order 9 has only straight lines for fundamental lines, both of the first and of the second kind. (Received October 30, 1931.)

405. Professor G. T. Whyburn: *On the decomposability of closed sets into a countable number of simple sets of various types.*

For any closed set C and any class K of closed sets, denote by $K(C)$ the set of all points x of C such that no neighborhood of x is contained in any K-set in

C. Set $C_1^k = K(C)$, $C_2^k = K(C_1^k)$, $\cdots$, $C_\alpha^k = K(C_{\alpha-1}^k)$ or $\prod_{\beta<\alpha} C_\beta^k$, according as α does or does not have an immediate predecessor, $\cdots$. We prove the following general theorem: A self-compact metric set C is the sum of a countable number of K-sets if and only if $C_\alpha^k = 0$ for some ordinal number α of the first or second class. An alternate form of this condition is that every subset N of C contain at least one point x for some neighborhood R of which the set $N \cdot R$ is a subset of some K-set in C. By taking K to be the class of all arcs, we have a solution of the well known problem of finding necessary and sufficient conditions that a given continuum be the sum of a countable number of arcs. Other choices of K are discussed, as are also the cases where C_n^k vanishes ($n = 1, 2, \cdots$) for general and specific K's. (Received October 24, 1931.)

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

COLUMBUS, OHIO, November 27-28, 1931.

By invitation of the program committee, Professor W. C. Graustein will speak on *Parallelism of Levi-Civita in classical differential geometry*, and Professor H. M. Gehman will speak on *Homeomorphic geometry*.

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, November 28, 1931.

By invitation of the program committee, Professor R. L. Moore will deliver an address entitled *A set of axioms for plane analysis situs*.

NEW ORLEANS, ANNUAL MEETING, December 28-31, 1931.

Abstracts must be in the hands of Associate Secretary Tomlinson Fort, 501 West 116th St., New York City, not later than November 28, 1931. Abstracts received by November 5 will appear in the November issue of the Bulletin. Professor E. R. Hedrick will deliver his retiring presidential address on the topic *Non-analytic functions of a complex variable*. At joint sessions with the American Physical Society, the ninth Josiah Willard Gibbs Lecture will be delivered by Professor P. W. Bridgman of Harvard University, on the topic, *Statistical mechanics and the second law of thermodynamics;* Professor G. A. Bliss will deliver his retiring vice-presidential address for Section A of the A. A. A. S. on the topic, *The calculus of variations of the quantum theory;* Professor W. F. G. Swann will read a paper by invitation on *The significance of the fundamental concepts of modern atomic theories;* and Professor George D. Birkhoff will speak by invitation on the topic, *Stability and instability of physical systems*.

The usual February Meeting in New York City will be omitted.

NEW YORK CITY, March 25-26, 1932.

Abstracts must be in the hands of the Associate Secretary Tomlinson Fort, 501 West 116th St., New York City, not later than March 5. All such abstracts will appear in the March issue of the Bulletin.

CHICAGO, April 8-9, 1932.

Abstracts must be in the hands of Associate Secretary Mark H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than March 18. All those received on or before March 6 will appear in the March issue of the Bulletin. By invitation of the program committee, Professor R. L. Wilder will deliver an address entitled *Point sets in three and higher dimensions and their investigation by means of unified analysis situs*.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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